Ernest Martin

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CALL FOR PAPERS

Studia Logica intends to devote its entire issue No. 2 or 3 of Vol. 37 (1978) to the topic: Relevance Logic. Articles and reviews touching on any aspect of the concept of relevance as it appears in logic are invited, with special emphasis on the tradition founded by Wilhelm Ackermann's concept of Strenge Implikation.

Papers should be sent to the guest editor for this number, Nuel D. Belnap, Jr., Department of Philosophy, University of Pittsburgh, Pittsburgh, PA 15260.

The deadline is Nov. 30, 1977. Because of definite size limitations, no paper should be more than 25-30 typewritten pages.

The following is quoted from a recent issue of Studia Logica: Manuscripts should be written in English. All manuscripts for publication should be typewritten with wide margin and wide spacing between the lines and should be sent in two copies. The Editors will arrange them for critical refereeing before acceptance. The authors should keep a complete copy of their manuscripts. Proofs will be sent to authors for correction. As a rule, the author will be submitted a proof of contribution for revision only once. The Editors assume the manuscript to be final version, any changes during galley-proofs are inadmissible. Authors receive 20 reprints of their papers free of cost.

THE SYSTEM RS

AN AMENDATION OF E

B.J. Copeland

The system RS (Restricted Substitution) formed part of my B.Phil. dissertation "Entailment Without Paradoxes: A Formal Approach" (University of Oxford, 1976). RS was born of a scepticism towards various of the theses which Anderson and Belnap use to motivate E. Chiefly, the system aims to retain paradox freedom whilst admitting the theoremhood of $(P \vee Q) . \sim P \rightarrow Q$. This is achieved by jettisoning closure under substitution. Below I give an abstract of my reasons for wanting to amend E, followed by an exposition and discussion of RS.

Firstly, a note concerning the use of quotation marks. Double quotes have every function which they customarily have in non-technical English. In addition they are used as a metalinguistic device to form names of expressions. For example

"p.q" is a name of the following formula : p.q

"P.Q" is a name of the following schema : P.Q

Single quotes will be used only to enclose schematic letters or schematic formulae. The context 'P' is not to be understood as the quotation mark name of the sixteenth letter of the Roman alphabet, but as a schematic representation of the quotation mark name of any formula represented by the schematic letter "P". Similarly, 'P.Q' is a schematic representation of the quotation mark name of any formula represented by the following schema: P.Q. Thus, for example, single quotes and schematic letters representing propositional variables can be used to construct a formula which represents all formulae having the same form as ""p.q" entails "p"", namely, the following formula: 'P.Q' entails 'P'. All this said, quotes are omitted except where confusion is likely to result. Quasi quotes also are always suppressed.

Getting down to business, E is, I maintain, incomplete, and in particular the first degree fragment of E is incomplete. That is, there are true (more correctly, valid) first degree entailment sentences which are not theorems of E. When do I count a first degree entailment as true? - Just in case

(a) its consequent and antecedent are mutually relevant and

(b) the replacement of $\rightarrow$ by $\supset$ results in a truth table tautology. No blasphemy so far. However, contra Anderson and Belnap, it seems to me that Belnap's necessary condition for relevance in terms of variable sharing, is also a sufficient condition for relevance. (It is to be understood that the notion of relevance is being discussed only in so far as it applies to formulae of the language of the propositional calculus extended by $\rightarrow$ and arbitrary semantically indistinguished constant sentences.)

I have convinced myself of the truth of this claim, for first degree entailments anyhow, in the following way. If the premiss and conclusion of an argument are not relevant, then it will not be possible to sensibly put the argument forward in any real life polemical situation (an assertion which I take to be the ground motivation of the search for relevance analyses of entailment). Hence, if it is possible to find an exemplification of an argument form, and a context, such that the exemplification is an argument which it is sensible to put forward in the context, then the premiss and conclusion of the argument form must be relevant; and accordingly the corresponding entailment is true (assuming that the argument form corresponds in the obvious way to a truth table tautology). But many first degree non-theorems of E satisfying (b) above and Belnap's necessary condition for relevance, can be shown by this method to be true entailments. Indeed, I have constructed a sizeable battery of exemplifications of non-theorems of E and pleasantly natural contexts wherein the corresponding arguments are manifestly sensible.[1] (A notable recalcitrant is $p.\sim p \rightarrow \phi\ (p.q)$, where ϕ

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1. This procedure was suggested to me by section XIV of Jonathan Bennett's paper "Entailment" (Phil.Rev.1969).

is any truth functional context. However, this is hardly surprising; and, similarly, many theorems of E equally resist fleshing out into real life polemics - e.g. $p.\sim p \rightarrow p$.) I conclude that in the light of this, the sufficiency of the Belnap criterion seems very likely.

On the basis of these considerations, I advance the following truth conditions for first degree entailments: $P \rightarrow Q$ is true iff $P \supset Q$ is a truth table tautology and P and Q share a variable. Whence it follows that E is incomplete.

A true non-theorem of E of particular interest is of course the entailment sentence "horizontalising" the inference figure of (extensional) Disjunctive Syllogism, namely $(P \vee Q).\sim P \rightarrow Q$ (which is henceforward referred to as DSE). In the spirit of the test for relevance just discussed, I have constructed an argument (indeed, several arguments) of the form $P \vee Q, \sim P / Q$ which, in the context provided for it, plainly constitutes an intuitively flawless, pre-analytically valid piece of argumentation. Moreover, the disjunctive premiss of the argument (call it "A or B") demonstratively does not support the subjunctive conditional: "If it were not the case that A, then it would be the case that B". Accordingly, the argument is not a special case of inference by Disjunctive Syllogism where the disjunctive premiss is intensional. Consequently, I feel in a position to query Anderson and Belnap's denial of DSE. (Theyhave twice in print reserved the right to try to find something funny about any example like the one I have been discussing. However, I tender a personal guarantee that mine is fine and faultless.)

Moving on to my second region of disagreement, I want to discuss the question of whether entailment is best treated logically in the form of a semantical predicate, or of a connective which nests without semantic increment. Entailment sentences report the validity of inference figures. Consider the inference figure: $p/p \vee q$. To say that the premiss of the figure entails the conclusion is to say something about a particular pair of truth functional formulae. Two specific inscriptions are mentioned, and entailment is

asserted to hold between them. The following piece of picture writing displays graphically the fundamental logical form of the entailment assertion in question: "p" entails "p∨q". A first degree entailment sentence, on its most natural logical parsing, is a sentence of a metalanguage for the language of the truth functional calculus, and the particle "entails" functions as a predicate of the metalanguage attaching to names of object language sentences. This much is intuitive and, despite Grammatical Propaedeutic, I think incontravertible. However, it is more or less jejune for our purposes. The interesting issues are: is it coherent in a formal entailment system with the vocabulary and range of schematic letters of E, to treat → as a connective which nests without semantic increment; and, if so, are the resulting benefits sufficient to warrant this departure from the above natural view of entailment?

Anderson and Belnap are explicit on the benefits of avoiding semantical predicates: "to take entailment as a sentence-predicate means that one has to involve himself with all the bag and baggage of the Metalanguage, which as we know very well is . . . dreadfully complicated in its syntactic department" (Entailment, p.492). Fair enough: E is certainly smooth and simple syntactically. If there are no gains and plenty of losses to be made in keeping use and mention distinct, why not treat → as part of the object language, provided we are satisfied that we can do it coherently?

I have two points to make. Firstly: semantic ascent certainly does involve a considerable loss in simplicity. Nonetheless, there is a considerable gain to be made by taking entailment metalinguistically: namely, it enables us to obtain a well motivated paradox free system in which DSE holds (as I hope I establish below). That is my main point. Secondarily, though, I have my doubts whether → can figure coherently in the object language. Questions of coherence arise once we begin to interpret the formalism. The variables of E, we are told, are to be eliminable by English sentences (e.g. Entailment p.491); and we see from the syntax that → may nest without

restriction in wff. Problem: how is → to be interpreted? It is readily seen that assigning to it the relation of entailment results in incoherence. Nor can it be read as a semantically non-cumulative connective, for example, "———" entails "———" (this is in fact the interpretation given to the → of RS). Anderson and Belnap's proposed interpretation is the semantically cumulative connective that —— entails that ——. However, this proposal raises problems in the vicinity of propositional identity. If the formula "A→~(~A.~B)" is to be understood as saying that that A entails that ~(~A.~B) is it also to be understood as saying that that A entails that (A∨B)? I think it must. For as I understand the function of the "that-" subnector in English, if T is a sentence embedded in a sentence S by means of the "that-" subnector, the substitution of a sentence synonymous with T for T in S results in a sentence synonymous with S. (Anderson and Belnap certainly offer no alternative construal of the "that-" subnector). Thus, since "~(~A.~B)" and "(A∨B)" are synonymous, so are, on the proposed reading of →, "A→~(~A.~B)" and "A→A∨B". (That is, synonymous in the sense of meaning the same under their intended interpretation. This is a stronger property than that of formal inter-derivability). But this is an unwanted result. An entailment system is a formalisation of rules of inference. The former formula corresponds to the rule "from A, infer ~(~A.~B)", whilst the latter corresponds to the rule "from A, infer (A∨B)". Clearly, these two rules are distinct. But on the proposed reading of →, E has to treat them as one and the same rule. The formula of E which formalises the first rule, also formalises the second. Thus if the proposed reading is adopted, E just is not a satisfactory entailment system.

To add a note of moderation, I am not trying to make out that for sure the syntax of E renders it an incoherent system. I am merely doubting that as yet the right wangle has been found for coherently interpreting E as a system of entailment sentences.

The proposal is, then, to treat entailment metalinguistically. Before attempting to do so systematically, I want to discuss how the axioms of E fare under this resolution. If we are willing to take $\rightarrow$ as a semantically cumulative connective, we can view E as systematically formalising the rules of inference of the propositional calculus, first degree entailment calculus, second degree entailment calculus,[1] and so on, all at once — formulae of all these calculi belong to the language of E, and so entailment formulae of arbitrary degree are provable in E. Taking $\rightarrow$ as non-cumulative, however, we see that to systematically formalise the rules of inference of these calculi, we have to take the calculi one at a time, and construct an entailment system for each. The axioms of an entailment system for the propositional calculus, for example, will be sentences of a metalanguage for the propositional calculus, and accordingly these axioms cannot simultaneously be axioms of an entailment system for a calculus of first degree entailment sentences, which must be sentences of a meta-metalanguage for the propositional calculus (first degree entailment sentences being sentences of a metalanguage for the propositional calculus).

Bearing this in mind, we now see that some of the axioms of E look suspicious. Suppose we view E as an entailment system for the propositional calculus (although the following remarks apply mutatis mutandis to E viewed as an entailment system for any calculus of n^{th} degree entailment sentences). If we rid ourselves of use/mention mixes by turning use into mention where necessary by, for example, the use of quotes, we should then expect the axioms of E to be sentences of a metalanguage for the propositional calculus. E4, E5, E8, E9, E11 and E14 can indeed be read as such. However, E6, for example, is a sentence of a meta-metalanguage for the propositional calculus. E6 corresponds not to a principle of inference of the propositional calculus,

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1. That is, a calculus of entailments between first degree entailments.

but to a principle of inference of a calculus of first degree entailment sentences. It is clear that the real function of E6 is to do duty for a rule of inference of E itself. Accordingly, it should be elided from the axioms of E, and the inference rule for which it is a surrogate should be added to to the inference rules of E. Alternatively, E6 may be stated as a meta axiom of E. This is a technique used below in the system RS. Since entailment sentences "horizontalise" rules of inference, a bulk of the inference rules of a given entailment system may be stated as meta axioms for the system, provided the proof theoretic procedures of the system are emended in an appropriate way. The technique allows an elegant presentation of systems which would otherwise be unwieldy owing to the presence of an excessive number of primitive inference rules.

Those axioms of E which need to be promoted to meta axioms are E6, E10 and E13.

E is allergic to its new metalinguistic make-up in a second way. Let "A", "B" and "C" be sentences of some L (thus "A" etc. are sentence-constants, not schematic letters). Consider the sentence S: "A" entails""B" entails "C"". ""B" entails "C"" is a sentence of a metalanguage for L. To obtain a sentence of a meta-metalanguage for L by filling the blank in: —— entails ""B" entails "C"", the blank must be filled by a name of an expression of a metalanguage for L. S does not satisfy this requirement, and is a sentence of no member of the hierarchy of languages we are considering. What S appears to want to say is that a sentence of a metalanguage for L can be inferred from a sentence of L. But there is no way of coherently saying this. (Indeed, it seems to me that there is no reason why we should want to say it. Inferential relations hold only within a language). What S actually says, if we supply additional quotes to turn it into a well formed entailment sentence, is that a name of a sentence entails a sentence, viz: ""A"" entails ""B" entails "C"".

Several of the axioms of E promote an incoherency similar to that in S;

namely E1, E2, E3, E7 and E12. In RS, coherent analogues to each of these axioms are used. For example, E2 - the principle of transitivity - occurs as a meta-axiom of RS in the coherent form "''P' entails 'Q' and 'Q' entails 'R'' entails ''P' entails 'R''".

I shall now present and discuss a hierarchy of entailment systems RS_i, of which the system RS is the first member. Section 4 below, with the exception of the first four paragraphs, is probably best omitted on a first reading.

1. The Context "P→Q"

I shall write "P→Q" for "'P' entails 'Q'", where the schematic letters "P", "Q" represent formulae of some language. Accordingly, any formula formed by writing → between two formulae of the propositional calculus, for example, is a formula of a metalanguage for the propositional calculus. Thus, no use/mention gaffe is committed by such formulae. Unlike the formulae of E, these formulae mention rather than use the formulae between which entailment is asserted to hold. To give an example, "p→pvq" is a formula of a metalanguage for the language of the propositional calculus, and asserts that "p" entails "pvq".

I adopt the convention that in unbracketed formulae or subformulae, → has major scope, and ~,.,v are standardly ranked in scope.

2. The Language Hierarchy

We define inductively a hierarchy of languages M_0, M_1, . . . , where M_0 is the object language of a propositional calculus. For i>0, each M_i is a metalanguage for its immediate predecessor.

(α) M_0

Vocabulary: atomic propositional variables p,q,r . . .; connectives .,v,~; parentheses),(.

wff of M_0: (i) any atomic variable standing alone

(ii) any truth function of atomic variables.

No other string of symbols is a wff of M_0.

(β) M_n ($n \geq 1$)

Vocabulary: the entire vocabulary of M_{n-1}; the connectives $.,\vee,\sim$; parentheses; the operator $\rightarrow$.

wff of M_n: (i) any wff of M_{n-1}, followed by the sign $\rightarrow$, followed by any wff of M_{n-1}

(ii) any truth function of wff of M_n.

No other string of symbols is a wff of M_n.

3. Preliminary Definitions

3.1 P is a wff iff for some $n \geq 0$, P is a wff of M_n.

3.2 P is atomic iff P is a propositional variable standing alone.

3.3 P is truth functional iff either P is atomic or P is a truth function of wff of M_0.

3.4 P is simple iff P is of the form $Q \rightarrow R$, where Q,R are wff.

3.5 P is complex iff P is a truth function of simple wff.

3.6 The degree of a wff P is n just in case P is a wff of M_n. The degree of a wff is thus a measure of the nestedness of the formula with respect to the entailment operator $\rightarrow$.

3.7 The antecedent, or protasis, of a simple formula P, where P is $Q \rightarrow R$:

$$\text{protP}=Q$$

3.8

$$\text{prot}_i P \quad \begin{cases} i=o : \text{protP} \\ i=n : \text{protprot}_{n-1} P \end{cases}$$

where P is simple

3.9 The consequent, or apodosis, of a simple formula P, where P is $Q \rightarrow R$:

$$\text{apP}=R$$

3.10 $$ap_i P \quad \begin{cases} i=o : apP \\ i=n : apap_{n-1} P \end{cases}$$

where P is simple

3.11 P is pure iff

(i) P is simple

(ii) for all n less than the degree of P, $prot_n P$ is either simple or atomic

(iii) for all n less than the degree of P, $ap_n P$ is either simple or atomic.

Thus P is pure only if P is a wff of some $M_{n>0}$. If P is pure, the only symbols in P are propositional variables, parentheses, and $\rightarrow$. The pure wff of some $M_{n>0}$ are the formulae of that M_n analagous to the atomic wff of M_0.

3.12 P is a uniform substitution instance of Q iff Q is a wff of degree n and R is a wf subformula of Q of degree n-1 and P is the result of uniformly replacing every occurrence of R in Q by some other wff of degree n-1 (except where Q is a wff of degree 0, in which case "0" is written for every occurrence of "n-1" in the definition).

3.13 P/Q iff P and Q share a variable.

3.14 matP - the material transform of P - is the result of replacing every occurrence of $\rightarrow$ in P by $\supset$.

4. Substitution in wff of the Language Hierarchy

The following inference rule will be termed the principle of uniform substitution: from some P, infer Q, provided Q is a uniform substitution instance of P. This principle is not in general a valid rule of inference in the language hierarchy. Various wff which are true, on their intended interpretation, have uniform substitution instances which are false, on their intended interpretation. For example, the formula $(p.q \rightarrow r) \rightarrow (p.\sim p \rightarrow q)$ is true, whilst a uniform substitution instance of it, $(p.q \rightarrow p) \rightarrow (p.\sim p \rightarrow q)$, is

false (since it has a true antecedent and a false consequent). Accordingly, the principle of uniform substitution does not figure among the inference rules of the formal entailment system associated with any M_i, in the general treatment of this hierarchy of systems given below.

Can uniform substitution be rejected so lightly? It is hard to see what rationale could be given to a rejection of the appropriate analogue to it in propositional calculus, for example. The change in dummy variables which produces

$(p.q \supset p) \supset (p.\sim p \supset \sim q)$ from $(p.q \supset r) \supset (p.\sim r \supset \sim q)$, just cannot be objected to. Again, it is difficult to see how uniform substitution of schematic letters in schemata of E could reasonably be prohibited. Thus, if $A.B \rightarrow C \vdash A.\sim C \vdash \sim B$ were a theorem schema of E (which it is not), the first paradox could be derived in E from E4,→E, and this theorem with the appropriate change in schematic letters. However, the situation is different for formulae of the language hierarchy. As remarked in Section 1, above, in a wff of the form $P \rightarrow Q$, P and Q are each in a quotation context. Thus, in substituting into formulae of the hierarchy, we are in effect substituting under quotes. It is a general phenomenon that the truth of a sentence containing a quotation context depends on which expression is quoted, and substitution under quotes, even of synonyms for synonyms, is not always truth preserving. Thus, the principle of uniform substitution does not have much to be said for it in our formal entailment systems, whose wff contain quotation contexts. We need give no rationale to our rejection of the principle, over and above remarking that it can, in some instances, lead from true formulae to false ones.

A dislike of the semantically cumulative treatment of → used in E led us to introduce quotation contexts. The presence of quotation contexts provides ample reason for not accepting the principle of uniform substitution. As will be shown later, it is the fact that this principle is not

unrestrictedly applicable in the system RS which enables the Lewis independent proofs to be blocked in RS. I find it extremely pleasing that the simple and intuitively desirable retreat to the metalanguage leads to a well-motivated faulting of the Lewis proofs which does not involve a denial of the entailment DSE.

However, the formal entailment systems I set out below need some apparatus for substitution. To construct a suitable substitution rule the principle of uniform substitution must be restricted in an appropriate way. The task of doing so is simplified by the fact that we want a substitution rule for use just in axiomatic systems, and thus need consider only substitution into formulae which are true (under the intended interpretation of the system). The obvious constraint, then, is that application of the substitution rule within a system never produces a false formula. Evidently, the only way that uniform substitution can falsify a formula is by the removal of relevance where present in the substituens, or by the introduction of relevance where not present in the substituens. To satisfy the required constraint on substitution, then, we have to keep tabs on variable-sharing during substitution. I shall go on to consider how this is to be done.

Where $P \rightarrow Q$ is a first degree entailment, and $R \rightarrow S$ is a uniform substitution instance of it, it is clear that if P/Q, then R/S. Moreover, since we are considering substitution into true formulae only, all first degree substitution instances will be substitution instances of formulae whose antecedent and consequent do share a variable. Thus, in the case of first degree formulae, uniform substitution neither removes nor introduces relevance. Accordingly, the principle of uniform substitution holds without restriction for first degree formulae.

Complications begin at the level of second degree entailments. Suppose $P \rightarrow Q$ is a true second degree entailment and $R \rightarrow S$ is a uniform substitution instance of it. Firstly, it is clear that, as for first degree entailments,

if P/Q, then R/S (and, indeed, this will obviously be the case for entailments of any degree). However, uniform substitution into a second degree entailment can alter the relevance relations between antecedents and consequents of first degree subformulae of the entailment. Consider the simplest possible case where all of P,Q,R and S are simple formulae (see above, definition 3.4). It may be the case that protR/apR, but not (protS/apS). This is an unwanted result in substitution, since R may be true, whilst S, which fails the variable sharing condition, cannot be true; and thus R↣S may have a true antecedent and a false consequent, and accordingly may be false. For example, given the envisaged distribution of variables, R↣S is false whenever matP is a tautology (where R↣S is a uniform substitution instance of P↣Q). For since uniform substitution does not affect tautologousness, if matP is a tautology, then so is matR. Accordingly, matR is a tautology and R satisfies the variable sharing condition, so R is true, whilst S is false.

In view of this, we impose the restriction that R↣S is a satisfactory substitution instance of P↣ Q iff if protR/apR, then protS/apS (where R↣S is a uniform substitution instance of the true P↣Q).

So far, so good. We now have to consider the case where any or all of P,Q,R and S are complex formulae (definition 3.5). Again, the restriction is going to be that the variables in R↣S are not distributed such that R may be true whilst S cannot be true.

Firstly, I shall consider the case where if either of R or S is complex it is either of the form T.U, or of the form T∨U, or of the form ~T, where T and U are simple formulae. I shall proceed by cases, and consider how variables may be distributed such that R may be true whilst S cannot be true.

Under what arrangements of variables may R be true?

Case 1: suppose R is T.U. T.U is true iff T is true and R is true. T is true only if protT/apT; and U is true only if protU/apU. Thus R is true only if both protT/apT and protU/apU.

Case 2: suppose R is T∨U. T∨U is true iff either T is true or U is true.

Thus R is true only if either protT/apT or protU/apU.

Case 3: suppose R is ~T. R is true iff T is false. T is false iff either matT is not a tautology or not (protT/apT). Our concern is with the question: under which distributions of variables may R, in the general case, be true? To which the answer is: under distributions such that not (protT/apT).

Under what arrangements of variables cannot S be true?

Case 1: S is T.U. S cannot be true if not (both protT/apT and protU/apU).

Case 2: S is T∨U. S cannot be true if not (either protT/apT/or protU/apU).

Case 3: S is ~T. In the general case, S cannot be true if protT/apT. (Safety First!)

We want to formalise the requirement that if as a result of uniform substitution, variables are distributed such that R may be true, then they are not distributed such that S cannot be true. To do this, the predicate rel, applying to formulae of M_1, is defined.

rel F

F is simple : relF iff protF/apF

T and U are simple:

- F is T.U. : relF iff protT/apT and protU/apU
- F is T∨U : relF iff protT/apT or protU/apU
- F is ~T : relF iff not (protT/apT).

We impose the restriction that R↦S is a satisfactory substitution instance of P↦Q iff if relR, then relS, where R↦S is a uniform substitution instance of a true second degree entailment P↦Q. This provides a satisfactory formalisation of the above requirement; and incorporates the earlier restriction we placed on substitution where both R and S are simple formulae.

It remains to consider the case where R and S are truth functions of complex formulae. Using the predicate rel and the truth conditions for the truth functional connectives, the predicate REL, applying to formulae of M_1, is defined.

RELF

F is simple	:	RELF iff relF
F is T.U and T,U are simple	:	RELF iff relF
F is T∨U and T,U are simple	:	RELF iff relF
F is ~T and T is simple	:	RELF iff relF
F is T.U and one or both of		
T,U is complex	:	RELF iff RELT and RELU
F is T∨U and one or both of		
T,U is complex	:	RELF iff RELT or RELU
F is ~T and T is complex	:	RELF iff not RELT

We impose the restriction that R↦S is a satisfactory substitution instance of P↦Q iff if RELR, then RELS (where R↦S is a uniform substitution instance of some true second degree entailment P↦Q). This captures formally for arbitrary formulae R,S of M_1 the requirement that if R↦S is a uniform substitution instance of some true second degree entailment P↦Q, then R↦S is a satisfactory substitution instance of P↦Q iff if the variables in R↦S are distributed such that R may be true, then they are not distributed such that S cannot be true.

The predicate REL can be used to formulate the restrictions on substitution for entailments of arbitrary degree. The predicate REL_i ($i \geq 2$) is defined as follows.

$REL_i F$

(α) $REL_2 F$ iff F is a formula of degree 2 and

if RELprotF, then RELapF

(β) $REL_n F$ iff F is a formula of degree n and

if REL_{n-1}protF, then REL_{n-1}apF

Thus, if the second degree entailment P is a uniform substitution instance of some true second degree entailment Q, P is a satisfactory substitution instance of Q iff $REL_2 P$.

The term "restricted substitution instance" will now be defined. (The substitution rules for the hierarchy of entailment systems are stated in terms of this notion.) To make for complete generality, the predicate REL_1, applicable to formulae of degree 1, is first defined.

REL_1F iff RELF

Then we define:

P is a restricted substitution instance of a formula Q occurring in some system RS_i iff P is a uniform substitution instance of Q and REL_nP, where n is the degree of P.

It should be noted that if P is a uniform substitution instance of a simple formula Q, then if protQ/apQ, then protP/apP. Thus any restricted substitution instance P of a simple formula occurring in some system RS_i is such that protP/apP. Hence we do not need an additional clause in the definition to ensure that restricted substitution instances of simple formulae do have this property. (They must possess the property, of course, since if not (protP/apP), then P is false, by the Belnap relevance condition.)

It must be stressed that it is not the case that if P is a uniform substitution instance of some formula Q of M_i, then P may be validly derived by substitution from Q iff P is a restricted substitution instance of Q. That P be so, is a sufficient, but <u>not</u> a necessary, condition. To see this, consider the true second degree entailments,
$\sim(p\rightarrow q)\rightarrow\sim(p\vee q\rightarrow q)$ and $\sim(p\rightarrow(r\vee s))\rightarrow\sim(p\vee(r\vee s)\rightarrow(r\vee s))$. Clearly the second is derivable from the first by substitution. However, the second is not a restricted substitution instance of the first, since $REL_1\sim(p\rightarrow(r\vee s))$ but not $REL_1\sim(p\vee(r\vee s)\rightarrow(r\vee s))$. In the case of second degree entailments, coincidence between the class of restricted substitution instances of proper axioms and theorems of RS_2, and the class of substitution instances which can be validly derived from them, can be achieved by modifying the original definition of rel in the clause for negation, taking into account whether F is the consequent or antecedent of a second degree entailment. Where F is an antecedent and F

is ~T, then relF iff not protT/apT (as before); but where F is a consequent and F is ~T, then relF (always). The reason I have not complicated the definition in this way is because with rel, REL_1, and REL_2 as I have defined them, the system RS, in which I am chiefly interested, is complete. The construction of a predicate REL_n such that it is a necessary and sufficient condition, as opposed to just a sufficient condition, for the derivability by substitution of P from some axiom or theorem Q of $RS_{n>2}$, that P is a restricted substitution instance of Q, is left as an exercise!

5. The Hierarchy of Entailment Systems

We define a hierarchy of axiomatic systems $RS_1, RS_2 \ldots$ by induction on degree.

(α) The system RS_1

(i) The proper axiom schemata of RS_1

The schematic letters P,Q,R represent atomic wff of M_0.

A1 $(P.Q)\rightarrow P$

A2 $(P.Q)\rightarrow Q$

A3 $P\rightarrow(P\vee Q)$

A4 $Q\rightarrow(P\vee Q)$

A5 $(P.(Q\vee R))\rightarrow((P.Q)\vee R)$

A6 $((P\vee Q).\sim P)\rightarrow Q$

A7 $\sim\sim P\rightarrow P$

A8 $P\rightarrow\sim\sim P$

A9 $P\rightarrow P.(Q\vee\sim Q)$

(ii) The meta axiom schemata of RS_1

The schematic letters P,Q,R represent atomic wff of M_0.

MA1 $((P\rightarrow Q).(Q\rightarrow R))\rightarrow(P\rightarrow R)$

MA2 $((P\rightarrow R).(Q\rightarrow R))\rightarrow((P\vee Q)\rightarrow R)$

MA3 $((P\rightarrow Q).(P\rightarrow R))\rightarrow(P\rightarrow(Q.R))$

MA4 $(P\rightarrow\sim Q)\rightarrow(Q\rightarrow\sim P)$

MA5 $(P \to \sim P \vee Q) \to (P \to Q)$

(iii) The inference rules of RS_1

The schematic letters of S,T represent wff of M_1.

R1 Any proper axiom of RS_1 may occur as a line of a proof. (It is axioms of RS_1, not axiom schemata, which may occur as lines of proofs.)

R2 If S occurs as a line of a proof, and T is a restricted substitution instance of S, T may occur as a subsequent line of the proof.

R3 If $S \to T$ is a meta axiom of RS_1, or a restricted substitution instance of a meta axiom of RS_1, and S occurs as a line of a proof, then T may occur as a subsequent line.

R4 If $S \to T$ is a proper axiom of RS_2, or a restricted substitution instance of such, and S occurs as a line of a proof, then T may occur as a subsequent line.

R5 If S occurs as a line of a proof, and T occurs as a line of the proof, then S.T may occur as a line of the proof subsequent to both.

(β) The system RS_n

(i) The proper axiom schemata of RS_n

The schematic letters P,Q,R represent pure wff of M_{n-1}.

[Each of the nine proper axiom schemata of RS_n is identical in form to the corresponding proper axiom schema of RS_1.]

(ii) The meta axiom schemata of RS_n

The schematic letters P,Q,R represent pure wff of M_{n-1}.

[Each of the five meta axiom schemata of RS_n is identical in form to the corresponding meta axiom schema of RS_1.]

(iii) The inference rules of RS_n

[Each of the five inference rules of RS_n is identical in form to the corresponding inference rules of RS_1, except that RS_n replaces

The systems of the hierarchy are identical in structure to one another. All that changes is the degree of the formulae represented by the schematic letters used in specifying each member of the hierarchy.

5.1 The restrictions on truth functional structure in axioms

As can be seen from R1 and the proper axiom schemata a proof in any RS_n commences from wff of M_n containing a minimum of truth functional structure, and such structure is introduced as required by substituting complex subformulae for pure subformulae, and truth functional subformulae for atomic subformulae, in accordance with R2. This makes for clear book keeping on relevance in the course of a proof. The same considerations motivate the restrictions on the schematic letters used in specifying meta axiom schemata.

5.2 The comparison with E

There is, of course, no axiom of E corresponding to A6. A6 ensures that the entailment DSE is an axiom of RS_1. Axiom E7 of E has no counterpart in the above axioms, since, for simplicity, I restrict my attention to non modal propositional calculus. Also the incoherent E12 has no counterpart, since there is no way at all, in the language hierarchy, of coherently expressing it. Apart from A8, A9 and MA5 (and A6) each of the above axioms and meta axioms corresponds in a clear way to some axiom of E. A8 is added because it seems to be independent of the other axioms. (It is derivable in E, I think, because of the presence of E12.) A9 (which receives some discussion in Section 8 below) is perhaps not the nicest of entailments to have as an axiom; but it is the simplest of a variety of formulae equivalent to it, one of which is required as an axiom if RS is to be complete (in the sense of page). In fact A9 corresponds to the incoherent E1 of E with all but the first occurrence of $\rightarrow$ read as $\supset$. Similarly, MA5 is required for the completeness of RS, and corresponds to the incoherent E3 of E with the second occurrence of $\rightarrow$ read as $\supset$.

R5 corresponds to the rule &I of E. R3 and R4 together do the work of →E of E. R4 reflects the fact that the proper axioms of any RS_n cannot be made to do double duty as inference rules of RS_n (whereas, because of semantic cumulativity, axioms of E can be thus employed). R4 could in fact be dispensed with if each of the proper axioms of RS_{n+1} were added to the meta axioms of RS_n. However, the method adopted is more economical. The method also exhibits the point that any RS_{n+1} provides a systematisation of the inference rules of RS_n. In fact, all the meta axioms of any RS_n, together with R3, could be dispensed with by adding the meta axioms of RS_n to the proper axioms of RS_{n+1}. However, if this procedure were adopted, RS_1 would have five less proper axioms than any other RS_n, thereby upsetting the symmetry in the hierarchy.

6. RS Defined

RS_1 achieves a systematised formalisation of the principles of inference of the propositional calculus (a claim which is substantiated by a completeness proof given later in the original dissentation). To arrive at such a formalisation is my chief concern, and the system RS_1 will be referred to simply as RS. Turning aside from our concern to emphasise the structural identity of members of the hierarchy of systems, we can make three improvements in the formulation of RS. The first two arise from the fact that restricted substitution and uniform substitution coincide for true first degree entailments (as noted in section 4). Firstly, we can replace "restricted substitution instance" in R2 by "uniform substitution instance". Secondly, we can allow the schematic letters used in specifying the proper axioms of RS, to represent <u>any</u> wff of M_o, and not just atomic ones. Lastly, we can make RS independent of RS_2 by adding each of the proper axiom schemata of RS_2 to the meta axiom schemata of RS. This then enables us to delete R4 from RS.

7. The Faulting of the Lewis Independent Proofs

Each of the Lewis proofs is conditional in structure, and cannot, therefore, be formulated in RS, which has no procedures for conditional proof. However, we can investigate the corresponding direct versions of the Lewis proofs in RS.

Consider the following attempt to prove the first paradox in RS by the moves used in the first Lewis proof.

1. $p.\sim p \rightarrow p$ A1
2. $p.\sim p \rightarrow \sim p$ A2
3. $p \rightarrow \sim\sim p$ A8
4. $(p.\sim p \rightarrow p).(p \rightarrow \sim\sim p)$ 3,1,R5
5. $p.\sim p \rightarrow \sim\sim p$ 4,MA1,R3
6. $\sim p \rightarrow \sim p \vee q$ A3
7. $(p.\sim p \rightarrow \sim p).(\sim p \rightarrow \sim p \vee q)$ 6,2,R5
8. $p.\sim p \rightarrow \sim p \vee q$ 7,MA1,R3
9. $(p.\sim p \rightarrow \sim p \vee q).(p.\sim p \rightarrow \sim\sim p)$ 8,5,R5
10. $p.\sim p \rightarrow (\sim p \vee q).\sim\sim p$ 9,MA3,R3
11. $(\sim p \vee q).\sim\sim p \rightarrow q$ A6
12. $(p.\sim p \rightarrow (\sim p \vee q).\sim\sim p).((\sim p \vee q).\sim\sim p \rightarrow q)$ 11,10,R5
13. $p.\sim p \rightarrow q$ 12,MA1,R3

Every step in the proof is satisfactory except the last. In moving from 12 to 13 in the way indicated, it is presupposed that the following is a restricted substitution instance of MA1: $(p.\sim p \rightarrow (\sim p \vee q).\sim\sim p).((\sim p \vee q).\sim\sim p \rightarrow q) \rightarrow (p.\sim p \rightarrow q)$ (call this formula T). However, it is not the case that $\mathrm{REL}_2 T$ and so T is not a restricted substitution instance of MA1.

It can be seen intuitively that T is false and hence not validly derivable by substitution into MA1. $p.\sim p \rightarrow q$ is false, since its consequent and antecedent do not share a variable. Thus, if the antecedent of T is true, T itself is false, since no true proposition entails a false one. But each of $p.\sim p \supset (\sim p \vee q).\sim\sim p$ and $(\sim p \vee q).\sim\sim p \supset q$ is a truth table tautology, and the consequent

and antecedent of each of the two corresponding entailments, respectively, share a variable. Accordingly, each conjunct in the antecedent of T is true (using the truth conditions for first degree entailments set out above). Thus, the antecedent of T is true, and so T itself is false.

The second Lewis proof fails in RS for a similar reason. To give an abridged direct version of the "proof" in RS:

1. $q \rightarrow (q.p) \vee (q.\sim p)$ theorem of RS

2. $(q.p) \vee (q.\sim p) \rightarrow q.(p \vee \sim p)$ theorem of RS

3. $q.(p \vee \sim p) \rightarrow p \vee \sim p$ A2

4. $(q.p) \vee (q.\sim p) \rightarrow p \vee \sim p$ 3,2,R5,MA1,R3

5. $q \rightarrow p \vee \sim p$ 4,1,R5,M1,R3

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It is readily verified that the move from 4 to 5 makes use of a substitution instance of MA1 which is not a restricted substitution instance.

8. Transitivity

This discussion of the failure of the independent proofs brings out the point that it is not the case that every uniform substitution instance of MA1 - the transitivity axiom - is a restricted substitution instance of MA1. Thus Transitivity - the principle that 'P entails Q' and 'Q entails R' together entail 'P entails R' - is not an unrestrictedly valid mode of inference in RS. No doubt a majority of logicians would take this fact as enough to show the inadequacy of RS. For the transitivity of the relation of entailment is usually taken to be one of its most fundamental and obvious properties. However, it seems to me undoubtedly true that Transitivity is not an unrestrictedly valid principle of inference. If used unrestrictedly, it gives rise to inferences from true premisses to false conclusions. An example in point is the inference of $q \rightarrow p \vee \sim p$ from $q.(p \vee \sim p) \rightarrow p \vee \sim p$ and $q \rightarrow q.(p \vee \sim p)$. The premisses of the inference are true, whilst its conclusion is false.

But, a critic might say, this position is surely odd. Can any justification

be given for taking $q \rightarrow p \vee \sim p$ as false, but $q \rightarrow q.(p \vee \sim p)$ as true? Surely relevance between consequent and antecedent has been gained too cheaply in the latter formula? Doesn't this phenomenon of the availability of "cheap relevance" simply show that variable sharing cannot plausibly be taken as a sufficient condition for relevance? Our critic's doubts are easily layed, however. Above it was canvassed that if it is possible to exemplify an argument form with an argument which can sensibly be put forward in some specified context, then the premiss and conclusion of the argument form must be relevant to one another. Given the inspiration, natural and convincing examples of sensible arguments of the form "q. Therefore $q.(p \vee \sim p)$" are readily forthcoming. Thus q and $q.(p \vee \sim p)$ are genuinely, not cheaply, relevant.

The liking which logicians have for the transitivity of entailment is no doubt partly explained by utilitarian considerations. Transitivity is an extremely valuable mode of inference; we should not get far in formal logic without it. (Indeed, practically every proof in RS itself relies at some point on MA1 and R3.) However, acknowledgement of the indispensibility of Transitivity engenders a liking for the principle that <u>every</u> instance of inference by Transitivity is valid only on the assumption that if entailment is nontransitive, there can be no way of determining syntactically whether a particular inference by Transitivity is, or is not, valid. But RS does have syntactical resources for determining just that. The inference of $P \rightarrow R$ from $P \rightarrow Q$ and $Q \rightarrow R$ is valid just in case $(P \rightarrow Q.Q \rightarrow R) \rightarrow (P \rightarrow R)$ is a restricted substitution instance of MA1.

B.J. Copeland

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RELEVANT ANALYTIC ENTAILMENT*

§1.

Let $\mathfrak{L}$ be a sentential language whose symbols are (a) denumerably many sentential variables $p, q, r, s \ldots p_1, q_1, r_1, s_1, \ldots$; (b) the logical constants $\wedge$ (conjunction), $\vee$ (disjunction), $\sim$ (negation) and $\rightarrow$ (entailment); (c) brackets and/or parentheses for punctuation. The well-formed formulas (wffs) of $\mathfrak{L}$ fall into two groups: (i) zero degree or truth functional wffs (zdfs) and (ii) entailments Entailments have the form

$$(A \rightarrow B)$$

where A and B are zero degree. The zero degree wffs make up the smallest set that contains the sentential variables and is closed under the operations of conjoining, disjoining and negating wffs.

The formal system characterized by the following fourteen axiom schemata and three rules I shall call $\mathfrak{E}$.[1]

Conjunction

A1 $A \wedge B \rightarrow B$

A2 $A \wedge B \rightarrow A$

R1 $A \rightarrow B$, $A \rightarrow C$, infer $A \rightarrow B \wedge C$

Disjunction

A3 $A \wedge (B \vee \sim B) \rightarrow A \vee B$

A4 $B \wedge (A \vee \sim A) \rightarrow A \vee B$

R2 $A \rightarrow C$, $B \rightarrow C$, infer $A \vee B \rightarrow C$

Excluded Middle

A5 $B \to A \vee \sim A$, provided every sentential variable occurring in A occurs also in B.

Negation

A6 $A \to \sim\sim A$

A7 $\sim\sim A \to A$

A8 $\sim(A \wedge B) \leftrightarrows \sim A \vee \sim B$

A9 $\sim(A \vee B) \leftrightarrows \sim A \wedge \sim B$

Distribution

A10 $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$

Transitivity

R3 $A \to B$, $B \to C$, infer $A \to C$

In stating the axioms and rules I have made tacit use of the usual conventions regarding deletion and insertion of punctuation symbols. The notation $\leftrightarrows$ occurring in A8 indicates that both $\sim(A \wedge B) \to \sim A \vee \sim B$ and $\sim A \vee \sim B \to \sim(A \wedge B)$ are axiom schemata - and similarly in the case of A9.

A <u>proof</u> (in $\mathfrak{g}$) is a finite sequence of entailments $(\gamma_1,\ldots,\gamma_n)$ where, as usual, each γ_i $(i = 1 \ldots n)$ is either an axiom or results from prior elements of the sequence of application of R1, R2 or R3. A proof $(\gamma_1,\ldots,\gamma_n)$ is a <u>proof of</u> γ_n and an entailment γ is a <u>theorem</u> of $\mathfrak{g}$ just in case there is a proof of γ in $\mathfrak{g}$. I shall write

$$\vdash_{\mathfrak{g}} \gamma$$

often omitting the subscript, to indicate that γ is a theorem of $\underset{\sim}{S}$.

$\mathbb{S}$ is interesting for several reasons. A glance at the axioms shows that every sentential variable that occurs in the consequent of an axiom occurs also in its antecedent, and it is clear that the rules preserve this "vocabulary sharing" property. So we have

Theorem 1.1. If $\vdash A \to B$, then every sentential variable that occurs in B occurs in A.

Thus none of the following entailments is provable in $\mathbb{S}$:

(1) $p \to q \vee \sim q$

(2) $q \wedge \sim q \to p$

(3) $p \to p \vee q$

(4) $\sim p \to \sim(p \wedge q)$,

and in this respect $\mathbb{S}$ resembles (the first-degree fragments of) the systems of Parry [5] and Dunn [4]. The latter possess the vocabulary-sharing property (in the sense of Theorem 1.1), but they also render

(5) $p \wedge (\sim p \vee q) \to q$

provable. (5) is not provable in $\mathbb{S}$ and so in this respect $\mathbb{S}$ resembles the system E_{fde} of Anderson and Belnap [1]. (3) and (4), however, are theorems of E_{fde}, and conversely

(6) $p \vee q \to q \vee \sim q$

is a theorem of $\mathbb{S}$ but not of E_{fde}.

If $A \wedge (\sim A \vee B) \to B$ (or the tidier $(A \wedge \sim A) \vee B \to B$) is added to the axioms of $\mathbb{S}$, then entailments such as

(7) $r \wedge (p \wedge \sim p) \to r \wedge \sim r$

can be proved, though (2) remains unprovable. So the decision to omit such an axiom amounts to a decision to divest inconsistencies of any of

their usual and special deductive significance. The closest one can come in $\mathbf{S}$ to a deductive "breakdown" in the presence of a contradiction is this : Let A be a conjunction of atoms (i.e., of variables or their negations) such that if a variable b occurs in A then both b and $\sim b$ occur as conjuncts of A; then $\vdash A \to B$ if and only if B is a truth function of the variables in A.[2] We note as something of a corollary to Theorems 3.2 and 4.3 below that $\mathbf{S}$ is nevertheless able to distinguish inconsistent zdfs from others: They are precisely those zdfs A such that $\vdash A \to \sim A$.[3]

The decision to include A5 is motivated by the need for a uniform treatment of disjunction. With the help of A5

(8) $\quad p \vee q \to (p \wedge q) \vee (p \wedge \sim q) \vee (\sim p \wedge q)$

can be proved, and the converse of (8) follows from A3, A4 and R2. Notice that (3) and (8) yield (1). Since (3) is - and (1) is not - a theorem of E_{fde}, (8) is not a theorem of E_{fde}. In contrast, $\mathbf{S}$ rejects (3) in favor of principles yielding the converse of (8). In this situation it is natural to retain (8) itself.

Although some conspicuous classical principles governing disjunction cannot be proved in $\mathbf{S}$, disjunctions retain most of their ordinary formal properties. This fact is recorded in the next few simple theorems. The first is a useful derived rule of inference.

Theorem 1.2. If $\vdash A \to B$, and every variable in C is in A, then $\vdash A \to B \vee C$.

Proof. If every variable in C is in A then by A5 $\vdash A \to C \vee \sim C$. Thus, if $\vdash A \to B$, then $\vdash A \to B \wedge (C \vee \sim C)$ (R1). So by A3 and R3,

$\vdash A \to B \vee C$.

Theorem 1.3.

(i) $\vdash A \vee B \leftrightarrows B \vee A$

(ii) $\vdash A \vee (B \vee C) \leftrightarrows (A \vee B) \vee C$.

(iii) $\vdash A \leftrightarrows A \vee A$

(iv) $\vdash A \vee B \leftrightarrows (A \wedge B) \vee (A \wedge \sim B) \vee (\sim A \wedge B)$

(The notation "$\vdash A \leftrightarrows B$" abbreviates the statement "$\vdash A \to B$ and $\vdash B \to A$." If $\vdash A \leftrightarrows B$, we say that A coentails B. The expression "$A \leftrightarrows B$" is called a coentailment.)

Proof. Observe first that we have as lemmas for the proof that $\vdash A \to A$, $\vdash A \wedge B \to B \wedge A$ and $\vdash A \wedge (B \wedge C) \leftrightarrows (A \wedge B) \wedge C$. I shall prove half of (i) and half of (iv) for illustration, leaving the remainder of the proof to the reader.

(i)

		Lemma
1.	$\vdash A \vee B \to A \vee B$	
2.	$\vdash A \vee B \to [(A \vee \sim A) \wedge (B \vee \sim B)] \wedge (A \vee B)$	1, A5, R1
3.	$\vdash [(A \vee \sim A) \wedge (B \vee \sim B)] \wedge A \to A \wedge (B \vee \sim B)$	A1, A2, R1
4.	$\vdash [(A \vee \sim A) \wedge (B \vee \sim B)] \wedge A \to B \vee A$	3, A4, R3
5.	$\vdash [(A \vee \sim A) \wedge (B \vee \sim B)] \wedge B \to B \vee A$	A1, 1.2
6.	$\vdash [(A \vee \sim A) \wedge (B \vee \sim B)] \wedge (A \vee B) \to B \vee A$	4, 5, R2, A10, R3
7.	$\vdash A \vee B \to B \vee A$	2, 6, R3

(iv)

1.	$\vdash A \vee B \to [(A \vee \sim A) \wedge (B \vee \sim B)] \wedge (A \vee B)$	Lemma, A5, R1
2.	$\vdash [\ldots]^* \wedge A \to (A \wedge B) \vee (A \wedge \sim B)$	A1,A2,R1,3.1(i), A10, R3

*[...] abbreviates $[(A \vee \sim A) \wedge (B \vee \sim B)]$

3.	$\vdash[\ldots] \wedge A \rightarrow (A \wedge B) \vee (A \wedge \sim B) \vee (\sim A \wedge B)$	2,1.2,1.3(iii)
4.	$\vdash[\ldots] \wedge B \rightarrow (B \wedge A) \vee (B \wedge \sim A)$	A1,A2,R1,1.3(i), A10,R3
5.	$\vdash[\ldots] \wedge B \rightarrow (A \wedge B) \vee (\sim A \wedge B)$	Lemma, 1.2,1.3(i), R2,4,R3
6.	$\vdash[\ldots] \wedge B \rightarrow (A \wedge B) \vee (A \wedge \sim B) \vee (\sim A \wedge B)$	6,1.2,1.3(ii)
7.	$\vdash[\ldots] \wedge (A \vee B) \rightarrow (A \wedge B) \vee (A \wedge \sim B) \vee)\sim A \wedge B)$	3,6,R2,A10,R3
8.	$\vdash A \vee B \rightarrow (A \wedge B) \vee (A \wedge \sim B) \vee (\sim A \wedge B)$	1,7,R3

Theorem 1.4. If $\vdash A \rightarrow B$ and $\vdash C \rightarrow D$, then $\vdash A \vee C \rightarrow B \vee D$.

Proof. Assume that $\vdash A \rightarrow B$ and $\vdash C \rightarrow D$. By Theorem 1.1, every variable in B is in A and every variable in D is in C. Since $\vdash A \vee C \rightarrow (A \wedge C) \vee (A \wedge \sim C) \vee (\sim A \wedge C)$, it remains to prove that $\vdash A \wedge C \rightarrow B \vee D$, $\vdash A \wedge \sim C \rightarrow B \vee D$ and $\vdash \sim A \wedge C \rightarrow B \vee D$ - each of which follows easily by applying Theorem 1.2 and (in the last case) Theorem 1.3(i).

The relation of coentailment in $\mathbb{S}$ is an equivalence relation (i.e., it is reflexive, symmetric, and transitive), but it lacks the replacement property. For example, although (8) and its converse are provable, the coentailment

(9) $\sim(p \vee q) \leftrightarrows \sim((p \wedge q) \vee (p \wedge \sim q) \vee (\sim p \wedge q))$

cannot be proved. Nonetheless, a very useful if not quite "standard" replacement theorem can be proved for $\mathbb{S}$. We turn now to this task.

Let

$$\vdash A \approx B$$

abbreviate the statement that $\vdash A \leftrightarrows B$ and $\vdash \sim A \leftrightarrows \sim B$. $\approx$ is an

equivalence relation, and it is clear that it possesses the replacement property. To prove this, we must first establish the following lemma.

Theorem 1.5. If $\vdash A \approx B$, then (i) $\vdash \sim A \approx \sim B$, (ii) $\vdash A \wedge C \approx B \wedge C$ and (iii) $\vdash A \vee C \approx B \vee C$.

Proof. (i) follows immediately from the hypothesis, A6 and A7. The conjunction axioms and the hypothesis jointly imply that $\vdash A \wedge C \leftrightarrows B \wedge C$, and A8 together with Theorem 1.4 and the hypothesis imply that $\vdash \sim(A \wedge C) \leftrightarrows \sim(B \wedge C)$, which proves (ii). (iii) is left to the reader.

Theorem 1.6. (Replacement) Let C_A be a zdf of which A is a well-formed part, and let C_B result from C_A by replacing zero or more occurrences of A in C_A by B. If $\vdash A \approx B$, then $\vdash C_A \approx C_B$.

Proof. (Sketch) Apart from the degenerate cases wherein either the replacement is at zero occurrences of A in C_A - in which case $C_A = C_B$ (and so since $\vdash C_A \approx C_A$, $\vdash C_A \approx C_B$), or $A = C_A$ - in which case $C_B = B$, the proof proceeds by induction on the construction of C_A. For example, if $C_A = (\sim D)_A = \sim(D_A)$ for some zdf D, then $C_B = \sim(D_B)$. By the induction hypothesis, $\vdash D_A \approx D_B$. So by Theorem 1.5(i), $\vdash \sim(D_A) \approx \sim(D_B)$.

Corollary. If $\vdash A \approx B$, then $\vdash C_A \leftrightarrows C_B$.

The next theorem paves the way for the reduction of entailments to entailments in normal form.

Theorem 1.7. If a coentailment $C \leftrightarrows D$ is a replacement principle, then $\vdash C \approx D$. A replacement principle is a coentailment having any one of the following forms:

1. (a) $A \wedge B \leftrightarrows B \wedge A$

   (b) $A \vee B \leftrightarrows B \vee A$

2. (a) $A \wedge (B \wedge C) \leftrightarrows (A \wedge B) \wedge C$

   (b) $A \vee (B \vee C) \leftrightarrows (A \vee B) \vee C$

3. (a) $A \wedge (B \vee C) \leftrightarrows (A \wedge B) \vee (A \wedge C)$

   (b) $A \vee (B \wedge C) \leftrightarrows (A \vee B) \wedge (A \vee C)$

4. (a) A9

   (b) A8

5. $A \leftrightarrows \sim\sim A$

6. (a) $A \leftrightarrows A \wedge A$

   (b) $A \leftrightarrows A \vee A$

Proof. (Sketch) It is easy though tedious to establish that any entailment having any of the forms listed above is a theorem of **S**. Then the theorem is first proved to hold of 4 so that the De Morgan principles may be exploited in conjunction with Theorem 1.6 in proving 2 and 3.

Following the usage of [1] an atom is to be a sentential variable or the negation thereof, and a primitive conjunction (primitive disjunction) is a conjunction (disjunction) of atoms. A zdf is in conjunctive normal form if it is a conjunction each of whose conjuncts is a primitive disjunction, and is in disjunctive normal form if it is a disjunction each of whose disjuncts is a primitive conjunction. Theorems 1.6 and 1.7 confirm that for each zdf A there is a zdf B in cnf and a zdf C in dnf such that $\vdash A \approx B$ and $\vdash A \approx C$. B (C) is said to be a cnf (dnf) of A.

If A is a zdf, $\varphi(A)$ is to be the zdf $(a_1 \vee \sim a_1) \wedge (a_2 \vee \sim a_2) \wedge \ldots \wedge (a_n \vee \sim a_n)$, where each a_i is a variable occurring in A and $a_1,\ldots,a_n$ lists all the variables occurring in A in alphabetic order and without repetitions. For example, if A is $\sim(\sim r \wedge q)$, $\varphi(A)$ is $(q \vee \sim q) \wedge (r \vee \sim r)$.

Theorem 1.8. $\vdash A \rightarrow B$ if and only if there is a dnf, $A_1 \vee \ldots \vee A_m$ of A and a cnf, $B_1 \wedge \ldots \wedge B_k$, of B such that for each i and j $(1 \leq i \leq m,\ 1 \leq j \leq k)$ $\vdash \varphi(A) \wedge A_i \rightarrow B_j$.

Proof. Since $\vdash A \approx (A_1 \vee \ldots \vee A_m)$ and $\vdash B \approx (B_1 \wedge \ldots \wedge B_k)$, $\vdash A \rightarrow B$ if and only if $\vdash A_1 \vee \ldots \vee A_m \rightarrow B_1 \wedge \ldots \wedge B_k$. Now $\vdash \varphi(A) \wedge A_i \rightarrow A_i$, and furthermore every variable in $A_1 \vee \ldots \vee A_m$ is in $\varphi(A) \wedge A_i$, so by 1.2, $\vdash \varphi(A) \wedge A_i \rightarrow A_i \vee (A_1 \vee \ldots \vee A_m)$. Thus $\vdash \varphi(A) \wedge A_i \rightarrow A_1 \vee \ldots \vee A_m$. So if $\vdash A_1 \vee \ldots \vee A_m \rightarrow B_1 \wedge \ldots \wedge B_k$, then $\vdash \varphi(A) \wedge A_i \rightarrow B_1 \wedge \ldots \wedge B_k$, and therefore $\vdash \varphi(A) \wedge A_i \rightarrow B_j$. Conversely, suppose that $\vdash \varphi(A) \wedge A_i \rightarrow B_j$, for every i and j. Then $\vdash \varphi(A) \wedge A_i \rightarrow B_1 \wedge \ldots \wedge B_k$ for every i. Thus $\vdash (\varphi(A) \wedge A_1) \vee (\varphi(A) \wedge A_2) \vee \ldots \vee (\varphi(A) \wedge A_m) \rightarrow B_1 \wedge \ldots \wedge B_k$, and by a principle of generalized distribution easily proved in $\mathbb{S}$, $\vdash \varphi(A) \wedge A_1 \vee \ldots \vee A_m \rightarrow B_1 \wedge \ldots \wedge B_k$. Since $\vdash A_1 \vee \ldots \vee A_m \rightarrow \varphi(A)$ it follows that $\vdash A_1 \vee \ldots \vee A_m \rightarrow B_1 \wedge \ldots \wedge B_k$.

The question whether an arbitrary entailment $A \rightarrow B$ is provable in $\mathbb{S}$ is reduced via Theorem 1.8 to the question whether entailments having the form $\varphi(A) \wedge C \rightarrow D$, where C is a primitive conjunction and D is a primitive disjunction, are provable in $\mathbb{S}$. This result is exploited in Section 3 and refined in Section 4.

§2. Semantics

For the semantics we invoke many-values - specifically, the values 2, 1, 3 and 0, which in light of the discussion in Section 2 of [2] we label affirmed, denied, both (affirmed and denied) and neither (affirmed nor denied) respectively. A valuation is a mapping from the set of zero-degree wff's into the set $\underset{\sim}{4}(= \{0,1,2,3\})$ that respects the following matrices:

$\sim$	
0	0
1	2
2	1
3	3

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	2	3
3	0	1	3	3

$\vee$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	2	2
3	0	3	2	3

An entailment, $A \rightarrow B$, is $\mathbb{S}$-valid if and only if for each valuation v, if $v(A) \geq 2$, then $v(B) \geq 2$. I shall write

$$\models_{\mathbb{S}} A \rightarrow B,$$

often omitting the subscript, to indicate that $A \rightarrow B$ is $\mathbb{S}$-valid.

The following are some important if obvious facts about $\mathbb{S}$-validity and its relation to validity in standard two-valued sentential logic. For this purpose we define a Boolean valuation for $\mathbb{S}$ to be a mapping from the set of zdfs into the set $\{1,2\}$ that respects the usual Boolean matrices for $\wedge$, $\vee$ and $\sim$ (here 1 corresponds to F and 2 to T). Also, $(A \supset B)$ is to abbreviate $(\sim A \vee B)$, and

$$\models_{C} B$$

indicates that B is a two-valued tautology.

No values in $\underset{\sim}{4}$ are "designated" and $\mathbb{S}$-validity is defined only for entailments. The $\underset{\sim}{4}$-matrices have the property that no zdf invariably takes on any one value in $\underset{\sim}{4}$. In fact, there are valuations v and v' such that for any zdf A, $v(A) = 0$ and $v'(A) = 3$. (Just let v be the valuation that assigns 0 to every variable of $\mathfrak{L}$; and let v' be the valuation that assigns 3 to every variable of 2. Moreover, for any zdf A, and valuation v, $v(A) = 0$ just in case $v(a) = 0$ for some variable a in A. It follows that we have the semantic analogue of Theorem 1.1, viz that if $\models A \rightarrow B$, then every variable in B is in A.

Although no zdf invariably receives any one value, some zfds invariably fail to receive the value 1. These, as we now show, are precisely the two-valued tautologies.

Theorem 3.1. Let A be a zdf and let v be a valuation such that $v(A) \in \{1,2\}$. Then there exists a Boolean valuation $\underset{\sim}{s}$ such that $\underset{\sim}{s}(A) = 1$ or $\underset{\sim}{s}(A) = 2$ according as $v(A) = 1$ or $v(A) = 2$.

Proof. Let $a_1,\ldots,a_n$ be all the variables in A and let $\underset{\sim}{s}$ be any Boolean valuation with the property that if $v(a_i) = 1$, then $\underset{\sim}{s}(a_i) = 1$ and if $v(a_i) = 2$, then $\underset{\sim}{s}(a_i) = 2$. We show by induction on the construction of A that if $v(A) = 1$, then $\underset{\sim}{s}(A) = 1$ and if $v(A) = 2$, then $\underset{\sim}{s}(A) = 2$.

Case (i). A is a variable. The result then follows by the definition of $\underset{\sim}{s}$.

Case (ii). A is $\sim$B. By the induction hypothesis, if $v(B) = 1$, then $\underset{\sim}{s}(B) = 1$ and if $v(B) = 2$, then $\underset{\sim}{s}(B) = 2$.

There are two cases to consider since either $v(A) = 1$ or $v(A) = 2$. If $v(A) = v(\sim B) = 1$, then $v(B) = 2$. Thus $\underset{\sim}{s}(B) = 2$ and so $\underset{\sim}{s}(\sim B) = \underset{\sim}{s}(A) = 1$. Similarly, if $v(A) = v(\sim B) = 2$, then $v(B) = 1$, and so $\underset{\sim}{s}(B) = 1$. Thus $\underset{\sim}{s}(\sim B) = \underset{\sim}{s}(A) = 2$.

Case (iii). A is $B \wedge C$. Again there are two cases: If $v(A) = v(B \wedge C) = 1$, then either $v(B) = 1$ or $v(C) = 1$. Hence, by the induction hypothesis, either $\underset{\sim}{s}(B) = 1$ or $v(C) = 1$. In either case, $\underset{\sim}{s}(B \wedge C) = 1$. If $v(A) = v(B \wedge C) = 2$, then $v(B) = 2$ and $v(C) = 2$. Thus $\underset{\sim}{s}(B) = 2$ and $\underset{\sim}{s}(C) = 2$. Therefore, $\underset{\sim}{s}(B \wedge C) = \underset{\sim}{s}(A) = 2$.

Case (iv). A is $B \vee C$. If $v(A) = v(B \vee C) = 1$, then $v(B) = 1$ and $v(C) = 1$. So $\underset{\sim}{s}(B) = 1$ and $\underset{\sim}{s}(C) = 1$. Thus $\underset{\sim}{s}(B \vee C) = \underset{\sim}{s}(A) = 1$. If $v(A) = \vee(B \vee C) = 2$, then either $v(B) = 2$ or $v(C) = 2$. Hence, either $\underset{\sim}{s}(B) = 2$ or $\underset{\sim}{s}(C) = 2$, and so $\underset{\sim}{s}(B \vee C) = \underset{\sim}{s}(A) = 2$.

Theorem 3.2. $\models_C B$ if and only if there is no valuation v such that $v(B) = 1$.

Proof. Since the $\underset{\sim}{4}$-matrices are entirely Boolean where only 1's and 2's are involved, every Boolean valuation is a valuation. So if no valuation gives B the value 1, no Boolean valuation does so either, i.e., every Boolean valuation gives B the value 2 - which makes B a auttology. On the other hand, suppose that some valuation v is such that $v(B) = 1$. Then according to Theorem 3.1, there's a Boolean valuation, $\underset{\sim}{s}$, such that $\underset{\sim}{s}(B) = 1$. In that case B cannot be a tautology.

Corollary. If $\models_S A \rightarrow B$, then $\models_C A \supset B$.

Proof. If $A \supset B$ is not a tautology, then by Theorem 3.2 there's a valuation v such that $v(A \supset B) = v(\sim A \vee B) = 1$. But then $v(A) = 2$ and $v(B) = 1$, so $A \rightarrow B$ cannot be S-valid.

The next theorem is the semantic analogue of A5.

Theorem 3.3. If A and B are zdfs such that every variable in B is in A, and $\models_C B$, then $\models_S A \rightarrow B$.

Proof. Suppose that $A \rightarrow B$ is not S-valid, i.e., that there is a valuation, v, such that $v(A) \geq 2$ and $v(B) < 2$. Since $\models_C B$, $v(B) \neq 1$. But if $v(B) = 0$, $v(a) = 0$ for some variable a in B, and therefore $v(A) = 0$ (since a is in A). So $v(B) \not< 2$ after all.

Finally, we note this rather unsatisfying fact - which represents an attempt to perceive how S-validity decomposes into classical validity plus a few further conditions:

Theorem 3.4. If $\models_C A \supset B$ and every variable in B is in A, then $\models_S A \rightarrow B$ if and only if there is no valuation v such that $v(A) = 3$ and $v(B) = 1$.

Proof. The proof is straightforward, using Theorem 3.2 and the fact that in view of variable sharing B receives the value 0 only if A does too.

§ 3. Soundness and Completeness

Most of the materials sufficient to establish that $\mathbf{S}$ is sound and complete relative to the semantics set forth in Section 2 are already at hand. In addition, we bother to note explicitly the following simple lemma.

Theorem 4.1. Let A be any zdf, B a conjunction $B_1 \wedge \ldots \wedge B_p$ and C a disjunction $C_1 \vee \ldots \vee C_q$ such that $\vdash B_i \rightarrow C_j$ for some $i = 1,\ldots,p$, $j = 1,\ldots,q$. If every variable in C is in $A \wedge B$, then $\vdash A \wedge B \rightarrow C$.

Proof. If $\vdash B_i \rightarrow C_j$, then $\vdash A \wedge B \rightarrow C_j$. Thus if C shares its variables with $A \wedge B$, $\vdash A \wedge B \rightarrow C_j \vee C$, and so $\vdash A \wedge B \rightarrow C$.

We proceed directly to the proof that $\mathbf{S}$ is sound and complete i.e., we show that $\vdash A \rightarrow B$ if and only if $\models A \rightarrow B$.

Theorem 4.2. (Soundness) If $\vdash A \rightarrow B$, then $\models A \rightarrow B$.

Proof. It is obvious that if $\vdash_{\mathbf{S}} A \rightarrow B$, then $\models_C A \supset B$. Moreover, since any theorem of $\mathbf{S}$ possesses the vocabulary-sharing property, Theorem 3.4 ensures that an axiom, $A \rightarrow B$, is $\mathbf{S}$-valid if there exists no valuation v such that $v(A) = 3$ and $v(B) = 1$.

Thus, let γ be an axiom of $\mathbf{S}$. There are seven cases to consider:

Case (i). γ is a conjunction axiom. Observe that if $v(A \wedge B) = 3$, then $v(A) \geq 2$ and $v(B) \geq 2$.

Case (ii). γ is a disjunction axiom. If $v(A \vee B) = 1$, $v(A) = v(B) = 1$. But $v(A) \geq 2$, if $v(A \wedge \sim B)) = 3$ and $v(B) \geq 2$, if

$v(B \wedge (A \vee \sim A)) = 3$.

Case (iii). γ is A5. See Theorem 3.3.

Case (iv). γ is A6 or A7. Note that $v(C) = 3$ iff $v(\sim C) = 3$, for any zdf C.

Case (v). γ is A8. Suppose $v(\sim(A \wedge B)) = v(A \wedge B) = 3$. Then $v(A) = 3$ or $v(B) = 3$. But if $v(\sim A \vee \sim B) = 1$, then $v(\sim A) = v(\sim B) = 1$ and $v(A) = v(B) = 2$. Conversely, if $v(\sim(A \wedge B)) = 1$, $v(A) = v(B) = 2$, so $v(\sim A) = v(\sim B) = v(\sim A \vee \sim B) = 1$.

Case (vi). γ is A9. Assume that $v(\sim(A \vee B)) = v(A \vee B) = 3$. Thus, either $v(A) = v(B) = 3$ or $v(A) = 1$ and $v(B) = 3$, or $v(A) = 3$ and $v(B) = 1$. But $v(\sim A \wedge \sim B) = 1$ only if $v(A) = 2$ or $v(B) = 2$. Conversely, suppose that $v(\sim A \wedge \sim B) = 3$. So either $v(A) = v(B) = 3$ or $v(A) = 1$ and $v(B) = 3$, or $v(A) = 3$ and $v(B) = 1$. But $v(\sim(A \wedge B)) = 1$ only if $v(A) = 2$ or $v(B) = 2$.

Case (vii). γ is A10. Assume that $v(A \wedge (B \vee C)) = 3$ and that $v((A \wedge B) \vee (A \wedge C)) = 1$. So $v(A \wedge B) = v(A \wedge C) = 1$ and $v(A) \geq 2$ and $v(B \vee C) \geq 2$. Hence, $v(B) \geq 2$ or $v(C) \geq 2$. In each case either $v(A \wedge B) \geq 2$ or $v(A \wedge C) \geq 2$.

It remains to verify that each of the rules of S preserve S-validity.

Since the rules preserve the property of being a tautology (with $\supset$ in place of $\rightarrow$) the corollary to Theorem 3.2 implies that if each premise of a rule is S-valid, then its conclusion is a tautology (again with $\supset$ in place of $\rightarrow$). And since the rules preserve vocabulary sharing, it suffices (by Theorem 3.4) to show that no valuation gives 3 to a and 1 to B if $A \rightarrow B$ is the conclusion of a rule each of whose premises is S-valid. If, in R1, $v(B \wedge C) = 1$, then $v(B) = 1$

or $v(C) = 1$. So if $v(A) = 3$, either $A \to B$ or $A \to C$ is not S-valid. As for R2: assume that $v(A \vee B) = 3$ and $v(C) = 1$. So either $v(A) = 3$ or $v(B) = 3$. Therefore, either $A \to C$ or $B \to C$ is not S-valid. Finally, consider R3 and suppose that $v(A) = 3$ and $v(C) = 1$. Then if $A \to B$ is S-valid, $v(B) \geq 2$. Hence, $B \to C$ cannot be S-valid. So either $A \to B$ or $B \to C$ must fail to be S-valid.

Theorem 4.2 now follows from the definition of proof in S.

Theorem 4.3. (Completeness) If $\models A \to B$, then $\vdash A \to B$.

Proof. Suppose it is not the case that $\vdash A \to B$. We shall argue that it is not the case that $\models A \to B$.

Since every zdf coentails an equivalent (in the sense of $\approx$) dnf and cnf, by Theorem 1.8 there is a dnf $A_1 \vee \ldots \vee A_n$ of A and a cnf $B_1 \wedge \ldots \wedge B_k$ of B such that it is not true that $\vdash \varphi(A) \wedge A_i \to B_j$ for some fixed $i.j$. Consequently, either (i) some variable in B_j is not in $\varphi(A) \wedge A_i$ or (ii) A_i and B_j fail to share an atom. (This is so because if $A_i = \alpha_1 \wedge \ldots \wedge \alpha_p$ and $B_j = \beta_1 \vee \ldots \vee \beta_q$ (where the α's and β's are atoms) share an atom, then $\vdash \alpha_{i'} \to \beta_{j'}$ for some $i' = 1,\ldots,p$, $j' = 1,\ldots,q$, and Theorem 4.1 applies.) In each case it follows that $\varphi(A) \wedge A_i \to B_j$ is not S-valid. For if (i) applies, it is clear that there is a valuation that gives B_j the value 0 and $\varphi(A) \wedge A_i$ the value 3. Now assume that every variable in B_j is in $\varphi(A) \wedge A_i$, and that A_i and B_j do not share an atom. Then B_j cannot be a tautology; for otherwise it would have to contain both b and $\sim b$ as disjuncts, for some variable b of $\mathcal{L}$. In that event A5 and Theorem 1.2 guarantee that $\vdash \varphi(A) \wedge A_i \to B_j$, contrary to our hypothesis.

Therefore, according to Theorem 3.2 there exists a valuation v such that $v(B_j) = 1$. It is now easy to construct a valuation v' such that $v'(B_j) = 1$ and $v'(\varphi(A) \wedge A_i) \geq 2$. We do so as follows: For any variable b of $\mathfrak{L}$,

(1) If b is in B_j, then $v'(b) = v(b)$; otherwise,

(2) If both b and $\sim b$ occur as conjuncts of A_i, then $v'(b) = 3$

(3) If b (but not $\sim b$) occurs as a conjunct of A_i, then $v'(b) = 2$

(4) If $\sim b$ (but not b) occurs as a conjunct of A_i, then $v'(b) = 1$

(5) If neither b nor $\sim b$ occurs as a conjunct of A_i, then $v'(b) = 3$.

Thus, $v'(B_j) = v(B_j) = 1$; and since $v'(c) > 0$, $v'(c \vee \sim c) \geq 2$, for each variable c of $\mathfrak{L}$. So $v'(\varphi(A)) \geq 2$. Furthermore, $v'(\alpha_{i'}) \geq 2$ - as we now show by cases: Case (i). $\alpha_{i'}$ is a variable. Suppose $\alpha_{i'}$ occurs in B_j. Then $\sim\alpha_{i'}$, but not $\alpha_{i'}$ itself (no sharing) occurs as a disjunct of B_j. So $v'(\sim\alpha_{i'}) = 1$ and $v'(\alpha_{i'}) = 2$. In each of the other relevant cases ((2) and (3)), $v'(\alpha_{i'}) \geq 2$. Case (ii). $\alpha_{i'} = \sim b$ for some variable b of $\mathfrak{L}$. If b occurs in B_j, it occurs as a disjunct since $\sim b$ does not. But then $v'(b) = 1$ and $v'(\sim b) = v'(\alpha_{i'}) = 2$. In each of the other relevant cases ((2) and (4)) $v'(\alpha_{i'}) \geq 2$.

We have shown that $\varphi(A) \wedge A_i \rightarrow B_j$ is not S-valid. Since it is easily seen that $\models \varphi(A) \wedge A_i \rightarrow A_1 \vee \ldots \vee A_n$, it follows that $A_1 \vee \ldots \vee A_n \rightarrow B_1 \wedge \ldots \wedge B_k$ is not S-valid. Finally, since $\vdash A \approx A_1 \vee \ldots \vee A_n$ and $\vdash B \approx B_1 \wedge \ldots \wedge B_k$, by the soundness theorem we have that $\models A \approx A_1 \vee \ldots \vee A_n$ and $\models B \approx B_1 \wedge \ldots \wedge B_k$. Consequently, $A \rightarrow B$ is not S-valid. This completes the proof.

§4. Concluding Remarks

Relevantly analytic entailments

In the spirit of [4] §15.2 we offer the following "perfect" interpolation theorem for $\mathcal{S}$.

Theorem 5.1. If $\vdash A \to B$, then there is a zdf C such that (i) $\vdash A \to C$, (ii) $\vdash C \to B$, and (iii) every variable in C is in both A and B.

Proof. Put $A \to B$ into normal form. For each A_i, let A_i^* be the least conjunction such that (a) $\varphi(B)$ is a subconjunction of A_i^* and (b) if α is an atom that A_i shares with some B_j, then α is a conjunct of A_i^*. (Note that A_i^* is always a non-null subconjunction of $\varphi(A) \wedge A_i$. Let C be $A_1^* \vee \ldots \vee A_m^*$. Then every variable in C is in B, and hence in A. So C satisfies (iii). Moreover, $\vdash \varphi(B) \wedge A_i \to A_i^*$, for every i. So $\vdash (\varphi(B) \wedge A_1) \vee \ldots \vee (\varphi(B) \wedge A_m) \to C$. Hence, $\vdash A \to C$. Finally, by Theorem 1.8, $\vdash \varphi(A) \wedge A_i \to B_j$. Therefore, by completeness, $\models \varphi(A) \wedge A_i \to B_j$. So either B_j is a tautology or A_i and B_j share an atom -- for otherwise we can construct an invalidating valuation for $\varphi(A) \wedge A_i \to B_j$ (as in the proof of Theorem 4.3). If B_j is a tautology, $\models A_i^* \to B_j$ (Theorem 3.3), and so $\vdash A_i^* \to B_j$. And if A_i and B_j share an atom, then again, $\vdash A_i^* \to B_j$ by Theorem 4.1. Thus $\vdash A_i^* \to B_j$, for every i and j. We may conclude that $\vdash C \to B$.

$\mathcal{S}$ is a Hilbert style formalism. The proof of Theorem 4.3 points the way to another sort of syntactic representation of the

class of $\mathfrak{S}$-valid entailments, one that may be for some purposes more convenient than $\mathfrak{S}$.

Let us say that an entailment $A \rightarrow B$ is relevantly analytic if and only if it satisfies the following two conditions:

(i) every variable in B is in A

(ii) $A \rightarrow B$ has a normal form such that for every pair of formulas A_i and B_j, either B_j contains both b and $\sim b$ as disjuncts, for some variable b of $\mathfrak{L}$, or A_i and B_j share an atom.

Using Theorem 1.8, it is not hard to show that an entailment is relevantly analytic if and only if it is provable in $\mathfrak{S}$.

Further developments

$\mathfrak{S}$ can be extended, more or less routinely, to a system $\mathfrak{SQ}$ incorporating a first-order quantification theory. Somewhat less routinely, the principles of relevantly analytic reasoning can be extended beyond the first degree to systems analogous to E and R of [4]. These matters are taken up in [3].

Harry Deutsch
1152½ N. Fuller Ave.
Los Angeles, CA 90046
June, 1976

Notes

*I wish to thank Jill Pipher for her many helpful comments on the subject of this paper.

[1]Remarks on various philosophical applications of the notion of entailment formalized by S may be found in [2].

[2]Given the results of §1, the proof of the "if" part is virtually the same as that given in ([1], p. 163) for E_{fde}. The "only if" part follows trivially from theorem 1.1.

[3]It also happens that A is a (two-valued) tautology if and only if $\vdash_S \sim A \rightarrow A$. So in a sense S contains classical propositional logic. E_{fde} lacks this property.

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RELEVANT ANALYTIC TABLEAUX* (ABSTRACT)

Michael A. McRobbie, Australian National University

Nuel D. Belnap Jr., University of Pittsburgh

Tableau formulations of classical propositional and predicate logic are an adaptation of a method used by Hintikka in [7] for constructing model sets and of the semantic tableaux of Beth [2], which themselves derive from the tree method used by Gentzen in [5] for proving theorems of his Sequenzenkalküle. These formulations were first studied by Smullyan in a series of papers ([11], [13]), and then received comprehensive treatment by him in [12]. Fitting [4] and Girle [6], among others, have studied tableau formulations of a wide range of modal logics (with necessity taken as primitive), and more recently Toledo [15] has given tableau formulations of type theory and other higher order theories. The use of tableau formulations of classical logics has been popularized by Jeffrey [8], Leblanc and Wisdom [9], and Rennie and Girle [10].

We adopt the nomenclature procedure of [1] and let TV stand for classical propositional calculus and $TV_{\supset}^{\sim}$ stand for the material implication/negation fragment thereof. We let TS be the tableau formulation of some arbitrary logic S formulated in the Hilbert style. Theoremhood in $TTV_{\supset}^{\sim}$ is defined as follows:

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DEFINITION 1. $\vdash_{TTV_{\sim}^{\supset}} A$ iff (i) a tableau can be constructed for $\bar{A}$ in $\sim^{\supset}$ which every branch closes once and only once.

This definition may be made more perspicuous if we remember that on the semantical level, some TV-material implication $A \supset B$ is true iff A is false or B is true. Hence in order to show that $\vdash_{TTV_{\sim}^{\supset}} A \supset B$ we show that a tableau can be constructed for $\overline{A \supset B}$ such that all its branches close, or in other words we demonstrate the mutual inconsistency of A and $\bar{B}$. But it should be noted that a tableau for $\overline{A \supset B}$ may close on some feature of A alone, e.g. A may be a contradiction, and hence a tableau for $\overline{A \supset B}$ will close solely because of features of A without $\bar{B}$ having had a rule applied to it, (where closure is considered as an explicit rule). Again, a tableau for $\overline{A \supset B}$ can also close on some feature of B alone, e.g. B may be a $TTV_{\sim}^{\supset}$-theorem, and hence the presence of A will be irrelevant in the closed tableau for $\overline{A \supset B}$.

The question then arises of what sort of a system we get if we impose a relevance condition on the classical definition of theoremhood in a tableau system as given above? More precisely, what system do we get if the requirements for the theoremhood of some arbitrary formula become not only satisfaction of condition (i) in Definition 1, but also satisfaction of a second condition which states that (ii) every formula appearing at a node in the tableau must have a rule applied to it at least once? The answer is precisely the implication/negation fragment of the Anderson-Belnap system R of relevant implication. So we

define theoremhood of a formula A in $TR_{\sim\rightarrow}$ as follows:

DEFINITION 2. $\vdash_{TR_{\sim\rightarrow}} A$ iff (i) a tableau can be constructed for $\bar{A}$ in which every branch closes once and only once and (ii) every formula appearing at a node in the tableau has a rule applied to it

since we can prove,

THEOREM 1. $\vdash_{R_{\sim\rightarrow}} A$ iff $\vdash_{TR_{\sim\rightarrow}} A$

It is also possible to prove that if we change condition (i) of Definition 2 so as to allow a branch to close more than once, we get a system which is precisely the implication/negation fragment of the Dunn-McCall system RM. Further, by placing restrictions on the use of the rule that allows us to break down formulas of the form $\overline{A \rightarrow B}$ we get the implication/negation fragment of the Anderson-Belnap system E of entailment.

A final point. This paper is not the first investigation of tableaux that are adopted so as to take into account relevant considerations. Dunn has shown in [3] that by putting a related relevance condition on the "coupled trees" formulation of TV as given by Jeffrey in [8], one obtains precisely the first degree relevant implications of R, (and hence of T and E).

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ULTRALOGIC AS UNIVERSAL?*

§1. A universal logic? A universal logic, in the intended sense, is one which is applicable in every situation whether realised or not, possible or not. Thus a universal logic is like a universal key, which opens, if rightly operated, all locks. It provides a canon for reasoning in every situation, including illogical, inconsistent and paradoxical ones. Few prevailing logics stand up to such a test. Certainly neither classical logic, nor the main alternatives to it offered, such as intuitionistic logic, are so universal. For they fail entirely in impossible situations. Moreover they are decidedly suspect even in apparently realisable cases, such as the empirically realisable situations of various semantical paradoxes and the situations of quantum physics - with the result in the latter case that it is sometimes suggested that classical logic is, like classical physics, only a good approximation to the empirically-selected and confirmed theory at the macroscopic level. The philosophical breakdowns or impasses when classical logic is insisted upon are much more widespread still. And these breakdowns count against a logic's claim to universality; for a universal logic should be adequate both for mathematical and philosophical purposes - and also for logical functions in other areas such as biology, economics, astrology, theology, and so on.

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* Caveat. The projects sketched out in this paper are in many cases programmatic, and in a very early stage of development. Since the procedures outlined often represent only a first attempt at seeing how an ultramodal theory would cope, the failure of given facets - by no means improbable - does not condemn the ultralogical program being proposed. Hardly necessary to say, failure of all likely approaches, or a suitable impossibility result, would condemn the program.

Though the ultralogical procedures sketched are mostly tentative I have avoided clogging the paper with appropriate qualifications, and, in the interests of feedback and of falsification of the conjectures, have often gone out of my way to state the theses in a bold and provocative way. Hopefully a more detailed and careful elaboration and defence of the various theses will be published subsequently (in collaboration with others). In fact much of the paper is a preview of [29]. But, because the paper is a survey one, it carries a heavy burden of references, particularly to work on relevant and dialectical logic published or to be published elsewhere.

I am aware of a heavy debt to several others to whom I owe some of the ideas tried out or through whose stimulation or assistance I arrived, separately or jointly, at the ideas tried out. Often too these others have arrived somewhat independently at the same ideas. I should mention, in particular, Ross T. Brady, J. Michael Dunn, Robert K. Meyer, and Valerie Routley. On the other hand, none of these others would, I suspect, approve, by any means, of all I propose, and I would not want to saddle them with the whole program. The relevant program can take many alternative routes to its foundational goal, and that I have chosen is only one, and perhaps an idiosyncratic one.

By selecting an ultralogic (i.e. an intensional logic which goes far beyond the modal[0] in the classifications it considers) as instrument of reasoning and argument assessment, very many of these gratuitous philosophical difficulties are avoided or resolved, in ways I will try to sketch out. And a relevant logic which does this can be appropriately universal. It can apply, without smudging all distictions, in impossible situations. It can apply in radically incomplete situations, and combined with intensional probability logic, which its use as a logical base dictates, it can deal with quantum anomalies. This too I will try to argue.

In short, an ultramodal logic can work everywhere. But it can work without serious logical loss. For, in particular, classical logic can be recovered in those situations (consistent and complete ones) where it is valid. Likewise other logics can be enthymematically recovered for the situations for which they do hold; and classical mathematics should be recoverable, insofar as it is correct.

These features - universality coupled with adequacy for the recovery of mainstream logics under appropriate conditions - provide initial support for the proposition that a relevant logic (with a suitably weak higher degree) would be a good choice as foundational logic, as the logic which is adopted in foundational studies, e.g. in the sciences, in linguistics and in mathematics. Indeed if such a logic really does formalise the central foundational relation of deducibility, as I believe, then it should be a best choice.

One would of course have to be wildly optimistic, and historically naive, to expect that anything is going to dislodge classical logic from its privileged position in foundational studies. It is too well-entrenched,[1] and too well-hedged around with defences. But this does not mean that one cannot see that there are better choices than those that are entrenched, and investigate to some extent the results of making an apparently better choice.

Nor is the idea of a universal logic some sort of transcendental illusion (as some have suggested the notion of a universal language or a universal science is). One source of the illusion claim is the view that one can never encompass in advance all situations. But generally logics proposed, notably first traditional logic and later classical logic, have claimed to be appropriately universal, to deal with all-comers among situations, to exhaust the cases (though what

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[0] A modal logic is, as usual, one where strict equivalents or provable material equivalents are intersubstitutable everywhere for one another preserving truth or provability. Classical logic is modal in this sense. An ultralogic is an ultramodal logic, primarily a logic which goes beyond the modal, where modal substitutivity conditions fail. But it is also supposed that an ultralogic has other desirable features, e.g. it includes a good implication, as well as a full complement of extensional connectives.

[1] It should be noted however how recently it became entrenched - only really since the second world war. Traditional logic had a vastly longer life span.

they really do is to rule out, as not situations, cases where they don't apply or where they break down). It is patent now that traditional logic failed rather conspicuously in its claim to be universal (though there are still defenders of the Aristotelian faith about, trying to extend the apparatus to cover previously written-off cases). It will become clearer, I hope, why classical logic is similarly, if perhaps not quite so conspicuously, inadequate. Why does any heir to classical logic in the historical chain - ultramodal logic, for example - stand a better chance of success?

The reasons to hope for success are, I think, of two connected sorts. The first concerns the way ultramodal logic applies, and the second concerns the intended interpretations of the central deducibility relation. Firstly, ultramodal logic applies in a reasoning situation c not by importing all its logical luggage into c, but through having situation c conform to its principles. Logical laws may fail in c (suppose, e.g., c is the set of Hegel's, or of some tribe's, beliefs). One draws out the consequences of what holds in c, e.g. of D in c, not by adding the thesis $D \rightarrow E$ to c, obtaining $D \& (D \rightarrow E)$ in c, and applying Ass, $D \& (D \rightarrow E) \rightarrow E$, in c to obtain that E holds in c. Rather one observes that the deductive consequences of D are obtained by closure of c under provable implications of the logic, so that where D holds in c and $\vdash D \rightarrow E$, E holds of c. Nothing says however that $D \rightarrow E$ holds of c; and it may indeed fail.

More generally, there is an important distinction many people are familiar with, but which classical and modal logics cannot draw, between a situation c's conforming to a law, of c's being lawlike, on the one hand, and of the law's holding in or belonging to c, on the other. At one extreme a situation may be lawlike though no logical laws hold in it (the null situation provides a degenerate example). Thus the application of ultramodal logic is not limited to consistent or to logically regular situations.[2] Its chances of success are thereby greatly enhanced.

Secondly, the central deducibility relation of ultramodal logics, entailment, is intended to capture the notion of sufficiency. This means, in particular, sufficiency of the antecedent of an entailment on its own, without any additional imported truths, especially imported logical truths. Thus entailment can work where an enthymematic implication cannot, because the imported truths may fail. Sufficiency is a go-anywhere notion, which is not limited by the fact that the situation in which it operates is somehow classically incoherent, e.g. inconsistent or paradoxical. If A is sufficient for B then it does not matter what else goes on; logical laws may go haywire but nothing subtracts from A's sufficiency. Incidentally this means also that, given A's sufficiency for B, A and anything else D is also sufficient for B. (Thus $A \& D \rightarrow A$ is correct, and

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[2] In [3] relevant sentential logic is enlarged so that it combines with all sentential connectives and caters for all situations. Observe however that it is not required, and would not be correct to require, that all set-ups the semantics considers are closed under relevant deducibility; non-deductive situations of course are not.

connexivism is eliminated as an option under the intended sufficiency interpretation.) Finally, since one is operating with a go-any-place logical notion one would expect general success.

It should be remarked in passing that logical sufficiency - as coupled with related intuitive models for deducibility such as those of total content inclusion and of containment - is what is really fundamental to the logics being promoted. Relevance of consequence to antecedent, though a hallmark of an adequate implicational relation, is strictly a by-product of a good sufficiency notion; for if B has nothing to do with A then A can hardly be sufficient for B. But relevance is not of the essence. (More technically, Belnap's weak relevance requirement, that there be no sentential theses of the form $A \rightarrow B$ where A and B fail to share a variable, is derivable from an inclusion account of entailment which models sufficiency.) The necessity requirement, that has been made much of by relevance and modal fans (cf. ABE), is likewise an outcome of a good sufficiency relation (as RLR again explains).[3]

It should also be remarked that the main early objectives of studies of relevant logics - of which at the usual sentential level ultralogics are special cases - were, firstly, to provide an analysis of entailment and its converse deducibility proper - and of the combination of entailment with other connectives, operators and constants, particularly truth-functional connectives and quantifiers - which met the criteria of relevance and necessity preservation; and, secondly and derivatively, to provide analyses of lawlike implications, i.e. non-necessary conditionals, and also of other conditionals. (The main results of these studies are assembled, or referred to, in ABE.) Ambitions have since vastly expanded, and now encompass relevant or ultralogical analyses of practically all the basic notions occurring in the foundations of mathematics and in the philosophy of science. It is of course in part the way that ultralogical analyses seem to solve problems left open - or created - by classical logic that has encouraged the universality proposal.

Ultralogic is thus being canvassed as universal. But a logic does not have to be universal. A logic (again like a key) can be designed to deal with a special class of cases, as were Łukasiewicz's $\vdash_3$ which was to cope with future contingents, Prior's tense logics to handle tenses, and so on (details may be found in [2]). But it is always supposed that these pieces fit into some grander design, e.g. in Prior's case into the larger framework of Russellian logic. In short, there are local logical theories, of this or that, but they do not rule out, but should mesh with, a more general logic.

It may be objected, however, against the ideal of a universal logic, that logics have to be local, that different sorts of situations have different sorts of logics. There is a logic for everyday situations (that's supposed to be classical logic), and there's a

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[3] The Kantian character of criteria proposed will not have passed unnoticed. Whereas Kant ([1], p. 27) proposed necessity and universality as sure tests of a priori knowledge and pure reason, universality and logical sufficiency are here being canvassed as conditions of adequacy for a satisfactory deducibility relation.

logic for microphysical situations (that's quantum logic of some brand), and maybe there's even a logic for impossible situations (that's presumably a dialectical logic). This option gets its extreme formulation in the slogan: 'Every sort of statement has its own sort of logic', which Wisdom (borrowing from Wittgenstein) thought was just part of the idiosyncratic platitude, but really represents an extreme parochialism. This local logics option soon runs into difficulties (as the geographical image suggests) at boundaries, as to how the local logics impinge upon one another and how they combine. For example what happens in a boundary area between two localities? In new (unclassified) situations? If one can't guarantee the location (e.g. because consistency isn't provable)?

Moreover some of the apparently local logic positions vanish when pressed into more global logics. For example, quantum and classical logics aren't really just locally related. Quantum logic is supposed to be universal, and classical logic is just a very good approximation, or some such, in classical physical situations.

Then there are some more theoretical difficulties about the local-logic picture. These concern the often remarked generality of logic, its scope of application, supposedly to all reasoning, and the fact that it is not limited by topic (its topic neutrality). How to turn these considerations into a convincing argument against logical parochialism is another matter (with which Haack [4], among others, has struggled). For the difficulties are supposed to turn on the formal features of logic; but progressively new local linguistic features, e.g. tenses, modifiers, adjectives, can be made issues of logical form. And there is no reason why several competing logical theories should not vie for a place, for example, as extending logics to cope with modifiers. These would in a good sense be local theories, but once again theories with the ability to hook onto, and really required to add to, a more general theory.

The local logic theme is sometimes supported by pointing to the range and diversity of logics these days. The argument is supposed to be that logics are just too diverse and heterodox for there to be a single canon of reasoning, or a universal logic. But the argument cannot be correct, as the universal semantics of [51], which organises all logics into a single semantical frame, with one deducibility relation, indicates. And, indeed, all connectives and quantifiers can be encompassed within the scheme of a universal logic of the sort proposed (along the lines of [3]): the trick is once again simply to allow for suitably many non-deductive situations.

The merely-local-logic thesis is likely to be reinforced by some higher powered (and accordingly wasteful) considerations drawn from limitative theorems. Surely a universal logic would have to be complete and finished: but this is an unlikely prospect at best, and limitative "theorems" now assure us that it is impossible (this is effectively Post's assumption and argument; see [34], pp. 395, 417). The short answer to this objection is that a universal logic does not have to be so complete, especially against classical codings into its syntax of semantical paradoxes or their like - indeed there are reasons we will come to for supposing that a universal logic ought to be incomplete, so as to reflect actual truth-value indeterminacy. What is more difficult to meet is the objection that if a logic is to

claim to be universal then it should be in principle completable - just as universal science should ultimately and in principle be able to encompass all scientific knowledge. That is a strong requirement, and it does not have to be conceded. It is probably enough for a universal logic that it be applicable to reasoning in every (deductive) situation: again like a key, it provides an organon. There need however be no such retreat: for the limitative results can be escaped, opening the way for completable logics and universal theories (see [59]).

Yet another objection threatens to overwhelm any universal logic project. The objection is that no logic can be universal because situations can always be found which fail to conform to any specific logical principles. But though such non-deductive situations can certainly be found (highly intensional functors will generate them, in terms of what holds where such a functor applies), it does not follow that a universal logic is thereby ruled out. There are two crucial requirements on a universal logic, but jointly they do not imply that a universal logic should apply within non-deductive situations. There is an important distinction between deductive, or logically controlled, situations and non-deductive situations; and the first requirement is that a universal logic should apply to all deductive situations; that is, all deductive situations should conform to the logic. For a deductive situation is one that is closed under deduction, i.e., syntactically, under provable entailment. But it is not the case that any old logic can be made out to be universal by appropriately restricting the class of deductive situations. For this class is independently, and naturally, determined. Inconsistent and paradoxical situations, for example, are commonly deductive situations which conform to requirements of reason; they cannot be arbitrarily, or as a matter of fact, ruled out as non-deductive. The second requirement is that a universal logic should allow for the logic and semantics of functors which determine non-deductive situations; for it should provide a logical framework for <u>all</u> functors. How these functors which pick out non-deductive situations can be dealt with logically is an important issue (taken up again in §4).

§2. <u>The relevant critique of extant logics, and especially of classical logic</u>. In brief the critique - which underpins the relevant case for new foundations - is as follows (much fuller, and less dogmatic versions of the critique, which include detailed discussion of positions outside the mainstream classical and modal positions on which I shall concentrate, may be found in ABE, RLR and [5]):- Firstly, these logics do not include an adequate theory of deducibility or its converse, entailment. No account of deducibility which contains the range of paradoxes of implication that the classical metalinguistic account and the corresponding modal systemic account admit, meets even minimal conditions of adequacy for a theory of deducibility. For deducibility as a sufficiency relation demands relevance.

Secondly, as an outcome of the first, these logics simply rule out proper logical examination of incomplete and inconsistent deductive theories, in particular of non-vacuous incomplete theories where not all logical laws hold and of non-trivial inconsistent theories where some contradictory propositions hold. According to modal logics, of which classical logic is a limiting case, there can be no such theories. A deductive theory is, as a matter of characterisation, closed under entailment. Hence such a theory, if non-vacuous, contains every logical law, by the paradoxical principle

that anything implies a logical law; so there can be no such incomplete theories. Similarly by the paradoxical principle that a contradictory pair entail everything, every such inconsistent theory is trivial; so there can be no such inconsistent theories. Yet plainly there are (and shortly we will encounter some), and one can, and sometimes must, reason about these theories deductively.

Thirdly, these logics preclude an adequate logical account of the intentional. For this reason they are philosophically inadequate; and their application to a wide range of problems in the philosophy of science has been (as we will see) disastrous. The success of classical logic in extensional areas, like parts of mathematics, cannot be repeated in the intensional sphere. The reason is that logical study of intensional notions requires the use of incomplete and inconsistent theories; such theories are beyond the reach of even the most liberal classical and modal semantics where all worlds admitted are both theorem-complete and consistent (see [14]). C.I. Lewis's modal treatment of the theory of propositions furnishes a simple example of the inadequacy of mainstream logics when it comes to intensional matters; for according to this theory there is just one necessary proposition and just one impossible proposition. The results of the application of mainstream positions to the logic of information, belief, perception, and so on, are equally appalling (as [6] argues in the case of belief).

Naturally classical theorists have a set of standard replies to such objections - one of the least satisfactory of which is to dismiss our everyday discourse, which is thoroughly intensional, from the purview of logic. It is said to be not really intelligible, or not worth bothering about because unscientific. These are pretty contemptible and easily met objections, but there are more cogent supporting reasons for trying to close out regions such as the intensional where classical logic fails. The sorts of reasons, which of course (circularly) appeal back to classical logic itself, will be familiar from the works of Quine, Goodman and others - intensional paradoxes, and so on. The ploys introduced, classical reshaping and formalising of mathematics, and amending[4] or closing off of areas of discourse to fit its theses, are typical strategems of an entrenched theory. So far these strategems are succeeding remarkably well with the plebs, especially in mathematics, one has to concede. However classical logic can be profitably compared with classical theories of art, e.g. of music; in time classical logic will be seen to be just as restrictive as classical form requirements in music.

It is not just that classical logic and its extensions are inadequate when applied outside the confined region of complete and consistent theories. Fourthly, classical logic and its extensions are plain wrong when so applied. For an integral part of classical logical theories is the rule of material detachment:

(γ) if A and $\sim A \vee B$ are theorems (true) so is B.

For suppose the deductive theory T we are studying is inconsistent but not trivial. Then (γ) wrongly trivialises it. For T, if

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[4] For example, by introducing transparent surrogates for opaque intensional functors.

inconsistent, will include p_o and also $\sim p_o$ for some formula p_o, and so by (γ) contains B for every B, that is, T is trivial. Nor is it just that we go wrong using (γ) in such cases, because (γ) like a paradox spreads inconsistency everywhere; (γ) cannot be reliably used in studying deductive theories, since these are not generally known to be negation consistent (this point is made in Łukasiewicz [7] and it is elaborated in RLR and [8]). For the general study of deductive theories a non-classical logic will have to be used. And only when consistency is established can (γ) be reliably introduced as an admissible rule: the proper role of (γ) should then be that of a derived rule (like the Cut rule it resembles), which becomes available when negation consistency is appropriately guaranteed. Thus too classical theories whose consistency can not be so guaranteed - and this includes all stronger theories - should be reformulated non-classically. The range in which classical logic can be reliably applied is accordingly very small, and classical logic is not wrong in a merely local way.

To sum up, the relevant rejection of other logics is based primarily on the following considerations:- Firstly none of the rivals captures the fundamental logical notion of sufficiency through which deducibility is characterised, and from which other hallmarks of a good entailment relation derive, e.g. relevance, preservation of containment features, and avoidance of suppression of necessary premisses. Secondly, the rivals are wrong and fail to conform to the facts, in particular, all the current rivals as foundational systems go wrong through their treatment of negation and consistency and the resulting incorporation of the rule (γ) of material detachment.

§3. The choice of foundations, and the ultramodal program. To try to show that ultramodal logic is a better choice as a general foundation (i.e. a foundation for all studies, not just mathematics) than classical logic for example, ultramodal logic has to be put through its paces. For it has to be pointed out in detail how ultramodal logic is, overall, a better choice than classical. This intuitive characterisation of the basis of choice of a general foundational logic can be considerably sharpened by way of multiple factor model for the choice of best objective (as expounded in [13]), according to which the best choice maximises on the weighted sum of factor values subject to a set of constraints (a most important constraint being conformity with the facts), and a better choice is one which results in higher values of the constrained weighted sum. One highly weighted factor is scope, the scope of a logic being a matter of the range of situations to which it can apply. A universal logic has maximal scope, but a logic like classical logic has only fairly limited scope. But scope is not the only factor, adequacy to the data is another important factor, and there are several pragmatic factors of non-negligible weight, such as simplicity, intelligibility, fruitfulness in applications, and strength.

A main strategy of the argument for ultramodal foundations will be to argue that ultramodal logic is far ahead of classical logic on scope and adequacy-to-data factors and does not lose out on pragmatic criteria, for a range of reasons; e.g. in the case of simplicity because the set theory furnished is, as a matter of inspection, simpler than classical alternatives (even if proofs, at first, seem harder); with intelligibility because the underlying logic has a

classical extensional semantics and so is, formally at least, no less unintelligible than its classical opponent; with fruitfulness and strength because what is correct in classical logic can be represented in it enthymematically.

To make good the initial point that ultramodal logic is far ahead on crucial factors, ultramodal accounts of several critical and representative foundational topics will have to be presented, in substantial detail. And this sets the course for the ultramodal program of research and classical theory replacement.

The big, overarching program is of course to redo things its way, to redo the formalisation of mathematics, the foundations of science, and logical issues in the philosophy of science, in particular, in an appropriately relevant way. Such a program is only in its infancy - relevant sufficiency logic itself is really only 20 years old, but is about to become of age. Moreover the program already faces some embarrassing difficulties;[5] for example, it is embarrassing that Peano-style formulations of "relevant" arithmetic are irrelevant in that any arithmetic identity entails any true arithmetic identity, e.g. 3=5 entails 10=10 (see Meyer [15]). What this means is that a reinvestigation of the Peano postulates for arithmetic in the light of relevant insights is needed. For some of mathematical reasoning may prove to be enthymematic, allowing the positive suppression of necessary truths and negative suppression of logical impossibilities. That mathematics should turn out to contain enthymematic reasoning would not be surprising; mathematicians are often busy men who like to take short-cuts, and commonly prefer mere sketches of proofs to detailed elaboration and verification. However it is always taken for granted nowadays - what often enough proves wrong without the introduction of further postulates - that a full proof could be set out in some appropriate framework. But even if a reconstruction could be done in the framework of strict implication (which allows the sorts of suppression mentioned), this is not good enough according to relevant and ultramodal perceptions: one should in principle go on and work out the larger ultramodal versions which reveal just what logical premisses are being employed and suppressed.

The demand for ultramodal foundations for science and discourse can take stronger or weaker forms. A strong version of the ultramodal position takes the form; the logical foundations ought to be ultramodal, not classical. The presumptive idea is that this upstart, ultralogic, should displace classical logic as underlying logic in the formalisation of scientific theory and of discourse. But there are less ambitious alternative demands, which it would be rash to dismiss. One is simply that ultramodal foundations should be seriously considered as an option, and perhaps held in reserve, just as intuitionistic foundations for mathematics are. After all, ultralogic is on the way to making a critique of classical logic and classical mathematics just as intuitionism has done (hopefully a more cogent criticism). The critiques have several common features: they both focus on the role of negation and, to a lesser extent, disjunction. To be properly considered as options, they should be developed so their

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[5] How many of these can be ascribed to lack of cleverness time may, or may not, tell.

features can be seen. Another proposal which can be coupled with the first more modest proposal is that investigation of relevant foundations is a worthwhile activity, viable in its own right. These proposals do not say classical logic ought to be displaced: they probably say that we should look at the options before too dogmatically dismissing alternatives to the classical foundation. They ask that research and education on ultramodal options go on and be encouraged - not dismissed, or persecuted or discriminated against or looked down upon. There is no need for us to adjudicate between these stronger and weaker proposals in most of what follows.

The ultramodal program will include, in the longer term, the following projects:-

1. The ultramodal reconstruction, or better, straightening out, of higher order or untyped logics and of set theory. This plunges us at once into all the issues raised by the logical paradoxes. But other issues that do not arise in the classical case also appear, e.g. the condition on substitutivity of identity, the form of extensionality axioms in set theory, the matter of when a function really does depend on its arguments, and so on. To these issues we will turn, beginning with a quite radical approach to the logical and semantical paradoxes permitted by ultralogic but fairly automatically excluded by all textbook logics. It should be emphasised, however, that the uniform dialectical treatment of logical and semantical paradoxes to be advanced is not one that has to be adopted by exponents of ultramodal or relevant logics; it is simply a very natural alternative that semantics for relevant logics and ultramodal analyses both powerfully suggest.

The diagnosis of the semantical paradoxes, even if hardly pressing for classical mathematics, is an important matter for a comprehensive linguistic theory (cf. Post [16]). Thus a uniform analysis of the paradoxes leads onto a second project:-

2. The design of ultralogical foundations for linguistics. The ultramodal thesis is that textbook logics are unfit to furnish the deep logical structure of natural languages, but that ultralogics should furnish an adequate logical base. The thesis gains further support from ultramodal analysis of intensional functions such as belief, perception and assertion - where once again many modally-induced paradoxes are removed en bloc.

3. Ultramodal - and paradox-free - reanalyses of the main logically investigated topics in the philosophy of science, in particular, probability, lawlike connections, counterfactuals, confirmation, evidence and information. Ultramodal analyses in some of these areas will be sketched out.

4. Ultramodal semantics for such non-transmissible, psychological, functors, as belief, perception, knowledge and assertion, and ultralogical foundations for psychology.

5. Ultramodal formalisation of intuitive, unformalised, mathematics and its parts. This project, like the previous ones, is not without its difficulties. Even the advance to one of the first stages, relevant formulations of arithmetic, has, as remarked,

run into significant problems. And ultramodal analysis is an untouched field. There is no doubt, however, that these fields can be encompassed in ultralogical investigations in one way or another - at worst by invoking appropriate extra assumptions as was done by the logistic program, with such extra axioms as those of infinity and choice - but better by revealing the enthymematic character of modern formalisations of intuitive mathematics. This point may help explain some of the reasons for confidence that ultramodal logic will cope somehow with the formalisation of mathematics.

Recovering the bulk of intuitive mathematics - which is not classical, except insofar as recent classical logical reconstructions have pushed it in that direction - is one thing: establishing the ultramodal adequacy of any such formalisation is quite another and more difficult matter. A first and weakest requirement of adequacy of a formalisation, e.g. of ultramodal analysis, is that of non-triviality, i.e. of absolute consistency. This much the program has in common with Hilbert's program. But, as is widely recognised, non-triviality is no guarantee of correctness, and stronger conditions of adequacy can easily be devised, though verifying that a formalisation meets them may be arduous or impossible. One such requirement is, of course, that of relevance: there should be no theorems of the form $A \rightarrow B$, where A is irrelevant to B.

6. Ultramodal reinvestigations of the classical limitative theorems. For it remains at present unclear to what extent these classical results will extend to ultramodal formalisations of mathematics, especially given diagnoses of the semantical paradoxes which fall outside the compass of levels-of-language frameworks.

To reveal, in sharpest form, how unified and thoroughgoing (and, one hopes, penetrating) the ultramodal program is, let us begin with the deeper philosophical issues that motivate the whole program.

§4. The impact of ultralogic on philosophical problems: ultralogic as a universal paradox solvent. One of the main negative theses being advanced is that classical logic and its extensions have buggered-up much philosophy, especially philosophy of science, and generated many gratuitous philosophical problems, and that these problems can be resolved using ultralogic. Indeed the obvious or naive solutions to several philosophical problems have been abandoned, and discussion subverted, only because of attachment to classical logic and its offsiders. It is time to try to make good some of these large claims, and to show what ultralogic is good for philosophically. But much of the treatment which follows does not pretend to be other than mainly synoptic (fuller development of these topics is attempted in other publications, in particular RLR and [29]).

Something of the damage wreaked by a bad entailment relation such as some variety of strict implication or its metalogical analogue, L-implication, has been observed, and is documented elsewhere (e.g. ABE and [14]). The damage - which results from modal-type treatments of negation and consistency, and shows up semantically in the restriction to possible situations and consistent models - spills over into many other areas, into the foundations of mathematics and of metalogic,

into the theory of propositions, of meaning, of information, of evidence and confirmation, into epistemology, and into ethics and the foundations of value theory. Perhaps, indeed, the implicational paradoxes are the source of all significant philosophical paradox?

A surprising amount of evidence has already been accumulated which makes it tempting to float the general thesis that the cause of a great many, if not all, philosophical paradoxes can be located in the implicational paradoxes. For example, in [30] Goddard develops the provocative thesis that the paradoxes of confirmation derive from the paradoxes of implication. Shortly we shall see the way in which the logical and semantical paradoxes depend on the implicational paradoxes. And elsewhere (especially [14]) it has been shown how the implicational paradoxes generate a wide variety of other philosophical paradoxes.

But while the general thesis comes close to the mark there is a deeper explanation which explains both the implicational paradoxes themselves, the paradoxes that the implicational paradoxes are supposed to explain, and paradoxes, such as the paradox of analysis, that the implicational paradoxes do not appear to explain. This deeper explanation is a semantical one, according to which all these paradoxes are produced through the orthodox restriction of semantical analysis to the possible, and so of the corresponding logical analyses within the confines of the modal. Such is the main thesis of Beyond of the Possible [29]. A corollary of this thesis is that a logical theory which penetrates the possibility barrier satisfactorily is going to furnish solutions to a great many philosophical paradoxes.

The detailed argument for the thesis proceeds through a case by case study of philosophical paradoxes and puzzles. However, the essence of the argument in a great many cases takes the following lines:- Every modal functor Φ satisfies the following substitutivity conditions for each of its places: if A is strictly equivalent to B, i.e. in symbols $A \leftrightarrows B$ (or metalinguistically, if $\vdash A \equiv B$) then $\Phi[A]$ iff $\Phi[B]$, where $\Phi[\ ]$ indicates a given place in Φ. But a great many, indeed most, and most philosophically important, intensional functors do not satisfy the condition. Entailment provides but one simple example. Since $A \& \sim A$ is strictly equivalent to $B \& \sim B$ - no possible worlds can distinguish them - if entailment, $\rightarrow$, were modal $A \& \sim A \rightarrow B$ would hold iff $B \& \sim B \rightarrow B$ should. But the latter holds, so, on the modal account, $A \& \sim A \rightarrow B$ - a familiar paradox. Examples are easily multiplied: if Hegel believed any contradiction he believed every contradiction; if it is desired to prove some necessary truth, it is desired to prove every, or any, necessary truth, and on the modal account a proof of any one would do; if a black raven confirms "all ravens are black" it confirms "All non-black ravens are ravens and not ravens" and vice versa; and so on. To press the point, and at the same time spell out the extraordinary ravens case, it is almost enough to quote Hempel ([48], pp. 11-12):-

> One remarkable consequence of this situation [application of the Nicod criterion combined with logical equivalence conditions] is that every hypothesis to which the criterion is applicable - i.e. every universal conditional - can be stated in a form for which there cannot possibly exist any confirming instances. Thus, e.g., the sentence

$$(x)\ [\text{Raven}(x)\ \&\ \sim\text{Black}(x)\ \supset .\ \text{Raven}(x)\ \&\ \sim\text{Raven}(x)]$$

is readily recognised as equivalent to both S_1 and S_2 above [i.e., to both $(x)(\text{Raven}(x) \supset \text{Black}(x))$ and $(x)(\sim\text{Black}(x) \supset \sim\text{Raven}(x))$]; yet no object whatever can confirm this sentence, i.e. satisfy both its antecedent and its consequent; for the consequent is contradictory.

Of course when universal conditionals are properly reformulated, with a relevant conditional, this "remarkable consequence" vanishes entirely; for the cited logical equivalence depends crucially on a paradox of implication, $A \rightarrow B \leftrightarrow .\ A\ \&\ \sim B \rightarrow A\ \&\ \sim A$, which in turn relies on the paradox $A\ \&\ \sim A \rightarrow B\ \&\ \sim B$ and, worse, on $\sim(A\ \&\ \sim B) \rightarrow .\ A \rightarrow B$.

In general, so long as the restriction to the possible is insisted upon, as it is classically, there is no way around modal substitutivity conditions and the ensuing intensional paradoxes. But lift the unwarranted restrictions, as ultralogic does, and the paradoxes vanish. Exactly how, a couple of examples will, subsequently, illustrate in detail (see §11). These examples, those of content and of semantic information, have been selected with a view to leading directly into problems in the philosophy of science, in particular to problems concerning probability and confirmation (§12) and thence (§13) problems of quantum logic.

These examples all concentrate however upon only one of the important classes of intensional functors, namely upon <u>transmissible</u> or, as one might almost say, rational functors. A functor Φ is $\rightarrow$-transmissible in a given place if whenever $A \rightarrow B$ then either if $\Phi[A]$ then $\Phi[B]$ or if $\Phi[B]$ then $\Phi[A]$. Functors that are $\rightarrow$-transmissible in each place, that is are fully transmissible, yield especially easily to semantical analysis in an entailmental framework which goes out beyond the possible (full details of the analysis many be found in RLR). Also easily catered for within this framework, which allows only for deductive situations, are functors which are either $\rightarrow$-transmissible or $\leftrightarrow$-transmissible in each place, the <u>logical</u> functors, so to say. Confirmation, preference, and obligation are examples of such logical functors, e.g. if $A \leftrightarrow B$ and $C \leftrightarrow D$ then if A confirms C then B confirms D.

Contrasted with the logical functors, in good positivist fashion - but of at least as much philosophical interest - are the psychological functors, those such as belief, desire, and knowledge, for which transmissibility fails in one or more places. For the semantical analysis of such functors, inclusion of non-deductive situations is inevitable, and the logical analysis has to be complicated accordingly. How this is done, and how ultralogic deals with the paradoxes which the orthodox logical accounts of psychological functors generate, is explained in detail elsewhere - in the case of belief in [6], for preference and obligation in RLR, and for a comprehensive range of intensional functors in [29]. The emerging ultralogical semantical analyses are not only non-paradoxical, they are more realistic, for they can cope with the beliefs, fears, wishes, knowledge and communication of actual people, as distinct from "ideally rational believers", "epistemically perfect beings", "ideal receivers", and so on, invoked in the interests of maintaining classical and modal paradigms.

The interest in such analyses is not confined to philosophy. It extends to all areas concerned with the program of obtaining a comprehensive theory, including a semantical theory, of natural languages; analyses of intensional functors are a crucial part of any such theory. It extends to areas in psychology concerned with the explanation of highly unorthodox behaviour. And so on.

§5. A dialectical diagnosis of logical and semantical paradoxes. The most notorious philosophical paradoxes are no doubt the logical and semantical paradoxes; but it is not difficult to see how the implicational paradoxes - especially the principle, $A \& \sim A \rightarrow B$, of ex falso quodlibet, which spreads any contradiction in a theory so as to trivialise it - magnify the effect of any antinomy, so that logical control is lost. Yet the main classical objection to logical paradoxes just is this loss of control. The orthodox reason for not even considering the option of admitting the logical paradoxes, as what they appear to be, proofs, is given by Hilbert and Ackermann ([17], p. 151):

> It is not possible to tolerate these various contradictions by accepting as a fact the provability of certain mutually contradicting sentences. For as soon as we admit any two mutually contradictory expressions A and $\bar{A}$ as true formulas, the entire calculus becomes meaningless ... .

But this argument depends precisely on the paradoxes of implication. Thus solutions of logical paradoxes are not independent of resolutions of implicational paradoxes. Contradictions need not trivialise a non-classical theory which is free of implicational paradoxes (they do not in any case render a theory absurd or meaningless). All the orthodox argument establishes is that a dialectical set theory cannot be based on a classical logical theory where any contradiction, no matter how isolated, induces triviality through such spread laws as $A \& \sim A \rightarrow B$. A dialectical logic has to be non-classical.

Even without the paradoxes of implication the apparent problem of isolated antinomies of course remains - unless a dialectical position is embraced, and such contradictions are just accepted as part of the theory. Such a dialectical position does however become a live option for anyone charmed by the semantical analysis of relevant logics and of ultramodal functors. For in the semantics one just does look at non-trivial inconsistent and incomplete situations and theories. Sooner or later it occurs to one: maybe the actual situation is one of these: maybe it is inconsistent! And then several reasons for giving serious consideration to this ordinarily-reckoned crazy option begin to emerge, not least of them the arguments for such antinomies as the semantical paradoxes, and the appeal of the ideal of a uniform ultramodal solution of all philosophical paradoxes. Quite apart from this, there is a strong case for a uniform solution for logical and semantical paradoxes, i.e. a single solution which encompasses, with at most variations due to subject matter, the whole range of paradoxes, both known and predicted (the case is presented in [18] and also elsewhere, e.g. [19]). A dialectical diagnosis enables one to offer such a uniform and unified account in a particularly simple way.

The central thesis - other important theses we will come to - of a uniform dialectical solution to both the logical and semantical paradoxes is that there are, as the paradoxes themselves show, some assertions which are overdetermined, and inconsistent, in that both they and their negations are true and other assertions which are underdetermined, and incomplete, in that neither they nor their negations are true. The inconsistent assertion L_0, "This very statement is false", provides an example of overdetermination since both L_0 and $\sim L_0$ are true, and its mate $\tilde{L}_0$, "This very statement is true", provides an example of underdetermination, because without further assumptions neither $\tilde{L}_0$ are $\sim\tilde{L}_0$ is assigned value true (even though $\tilde{L}_0 \vee \sim L_0$ may hold). In short, on this naive dialectical position, the paradoxes are taken to demonstrate that what the semantical defence of relevant logics argues is so at least for remote situations - seemingly that there are both incomplete situations and also inconsistent situations - is not so remote at all and in fact obtains for the actual situation T, where truth is assessed. Thus the world T, construed as everything that is the case, is inconsistent on the singularities the paradoxes provide. There is nothing impossible or absurd about this (as [8], e.g., explains).

There is little doubt but that this dialectical solution offers a uniform "solution" to the whole range of paradoxes, logical or semantical. It simply admits the paradox arguments at face value, as establishing isolated inconsistencies. What it disputes is the usual presupposed logic which spreads the effects of these paradoxes to the rest of discourse. (That is, a <u>thoroughly</u> uniform solution deals both with the full range of logical paradoxes and with the paradoxes of implication.) Given the simplicity of the basic solution it is a straightforward matter to apply it to most paradoxes that arise, e.g. to all those of Prior's family of paradoxes [9]. For example, the man who, unbeknown to himself in room 7, thinks at 6 only that everything thought in room 7 at 6 is false, has only a paradoxical thought; for the content of his thought reduces to the thought that this very thought is false, i.e. to a variant of the liar paradox. The man is undoubtedly thinking, though paradoxically, since his thought is true and its negation is also true. But this paradoxical thought leads to no cataclysmic collapse; it is, exactly as it appears to be, relatively isolated. Indeed we can engage in paradoxical thoughts reflectively without putting the rest of the world out of joint or forging necessary connections with other supposedly separate statements (in the amazing way Prior envisages in [9]).

There is nothing new about a dialectical diagnosis of paradoxes. Aristotle's insistence that 'it is the most true that two contradictory judgements cannot both be true' did not win general acceptance in ancient times. In particular, the dogma was rejected by Heracleitus and by Antistenes, the cynic. Nor were the arguments by which Aristotle supported his consistency dogma by any means conclusive, and some of them were not even persuasive (as Łukasiewicz has demonstrated in [7]). Furthermore there remained outstanding cases of convincing arguments leading to contradictory conclusions, the ancient semantical paradoxes for example. In the early 19th Century Hegel, influenced by Heracleitus, challenged the consistency dogma and traditional logic, and proposed a different logical theory, though one with ancient roots, dialectics, according to which

contradictory judgments can jointly hold true. This proposal was absorbed in marxism, and has become fundamental in contemporary marxist philosophy. Precise and detailed logical investigation of dialectical logics was initiated by Jaśkowski, who (while retaining considerable reservations about the admission of contradictions outside the context of multi-party discussions) was keenly aware of the important role his discussive logic could play in giving a dialectical treatment of paradoxes (see e.g., [36]). Jaśkowski's researches have been advanced and considerably extended by the work of da Costa (see, e.g., [56]) and his co-workers in Brasil, by Asenjo and Tamburino [50], and by others. But the underlying logics adopted mostly remained, like Jaskowski's systems, excessively strong, and overly intuitionistic, for a natural treatment of the logical paradoxes.

All that is new then is the simple and natural amalgamation of dialectical and relevant insights within the framework of ultralogic.

How the dialectical position manages to get away with its brash central thesis is perhaps best illustrated in detail (as in §6) in the case of the set-theoretical paradoxes; for the basic axioms of the subject are more fully developed and better known than in the case of semantical paradoxes, for instance. But, first, it is important to look at the supplementary theses that can accompany a dialectical resolution. For the dialectical admission of the paradoxes as proofs and as establishing the inconsistency of the full set theory world is, naturally, not the whole story. A dialectical diagnosis of the paradoxes really needs to be filled out by various supplementary theses - in a way that makes it evident how more classical perceptions fit into a dialectical framework.

Firstly, the origin, character and mechanism, so to speak, of the paradoxes still has to be explained, and ideally some guide - it does not have to be effective - provided as to when inconsistency or incompleteness is likely to occur and which regions are safe, in the particular sense of negation consistent. Most of these demands can be met by a theory of content self-dependence (as developed in [18] and briefly sketched in [20], p. 172). The basic idea may be illustrated as follows: the sentence 'This statement is false' is content self-dependent (i.e. its putative content depends on its own content), not in all contexts (e.g. not where 'This statement' refers to some other independent statement), but in self-dependent contexts, i.e. in contexts where 'This statement' refers back just to, depends just on, the intended content of the very sentence in which it occurs. Because of this closed content dependence loop the assertion is

content self-dependent in the self-dependence contexts.[6] The closed dependence loop enables a switching of truth-value assignments to occur; for the dependence loop allows the truth-value of a given assertion to be that of its own negation. It allows incompleteness as well as inconsistency, since if the content and truth-value of an assertion depends in the end just on itself then the truth-value of the assertion will never be resolved in other than an arbitrary way.

More complex cases of self-dependence where a chain of dependence relations is involved are explained similarly. For example, in the case of the pair

(1) (2) is true

(2) (1) is false

the content of the second assertion (2) depends on itself though the closed dependence loop goes through (1). And again the self-dependence and resulting endless looping allows an inconsistency producing truth value switch to be imposed.

The dialectical diagnosis can adopt this sort of account of the genesis of the paradoxes without however encountering the usual serious disadvantages of self-dependence or self-reference style accounts which aim to somehow outlaw content self-dependent statements; namely it does not have to pretend, what is far from obvious, that there are effective methods for determining in advance when and where content self-dependence will occur, and, moreover, that the effective methods can be applied if not to rule out just the cases of content self-dependence and not a wide range of other perfectly admissible statements in the formation rules of an ideal language, at least in the

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[6] Thus, so it is claimed in [20], the sentence lacks content, and is statement-incapable, in these contexts. The same theses, that the paradoxes involve vicious content self-dependence and open reference loops, and accordingly that the paradox-generating sentences involved lack content and fail to yield genuine statements, were developed in [18], [22], and SC. But there are options as to how to state matters where content self-dependence occurs, and the line taken in [18] and [20] is only one of the options. The dialectical position is another, and apparently superior, option, since it does not impose unwarranted limits on the bounds of logical reasoning, and since the statement-incapability option maps into it, that is the main theses of the lacking-content resolution of the paradoxes can be expressed within the dialectical framework.

The content self-dependence resolution - according to which content self-dependent sentences do not have a genuine content, and in the case of indicatives, for example, do not yield statements - has been elaborated (though not to requisite point where logical systems and restrictions emerge) in Mackie's more recent [19]. The content self-dependence resolution is of course but a refinement, an important and needed refinement, of Ryle's namely-rider method - a resolution of the paradoxes which is to be found, in essence, in Meinong.

axioms of its accompanying logic. But such axioms, most conspicuously in case of set theory, remain to be produced: failure to produce such axioms cannot however be attributed, as in the case of effective formation rules, to the impossibility of the business, but perhaps just to failure of human ingenuity. The dialectical account has the tremendous advantage that it does not have to wait upon the production of such axioms, if they can be devised, but provides a logic and theory in terms of which such axioms may be sought, or their unavailability in suitable form demonstrated.

Before turning to a detailed study of dialectical set theory, there is a further stage in the dialectical process that we should glance at, namely dialectical ascent to, or incorporation of, the metalanguage. In particular, it is instructive to consider the way in which the dialectical solution can deal with the commonly presented informal sketches of Gödel's theorems. Consider a variant on Kleene's argument ([26], p.205), which is but a simplification of Gödel's original informal argument ([24], p.598). It is assumed, firstly, that one can find, by diagonalisation, a constant assertion A which says of itself, in the metalanguage, that it is not provable, i.e.

$$A \leftrightarrow \sim Prov\ A \qquad (1).$$

This takes up in part Kleene's first premiss, that A means that A is unprovable; for if A means this then it certainly implies it, and vice versa. Kleene's second assumption is the correctness assumption that false formulae are unprovable,

$$False\ A \rightarrow \sim Prov\ A \qquad (2).$$

or as Gödel puts it, taking it that what is not false is true, provable assertions are true, i.e. for every B, Prov B → True B, or, as both Kleene and Gödel assume in their arguments, and as follows using the elimination principle, True B → B,

$$Prov\ B \rightarrow B \qquad (2').$$

Now let us assume, thirdly, that not only does (1) hold but also, what substitution in (1) would provide were it permissible,

$$\sim A \leftrightarrow \sim Prov\ \sim A \qquad (3).$$

Classically such an assertion A cannot be obtained, without disaster, any more than a truth predicate can be defined for rich languages, but dialectically such an A is quite admissible: it is simply demonstrably inconsistent, as both it and its negation are true. The argument is as follows:-

1.	Prov A → ~Prov A	from (1) and (2')
2.	~Prov A	from 1 by DL principles
3.	A	by 2 and (1)
4.	Prov A ↔ ~Prov ~A	by (3) and (1)
5.	Prov ~A	by 2 and 4
6.	~A	from (3) and 5
7.	A & ~A	3 and 6.

Even without (3) the dialectical conclusion should not be the standard one, that A is undecidable but true - though it can be. For ~Prov A but A as above, and, assuming (2') generally, B → ~Prov ~B, whence ~Prov ~A. But (2') causes distortion dialectically in cases where both D and ~D are true and one is provable, in virtue of its consistency outcome B → ~Prov ~B; for then Prov D and ~Prov D - the inconsistency spreads to the proof-theoretic apparatus. What is correct, and avoids this spread, is the rule form of (2'): Prov B ⇒ B. Then however undecidability vanishes: there is nothing to prevent ~A's being true and provable (cf. for more detail [59]).

Truth as well as provability is, naturally, dialectically expressible, and will presumably satisfy the general condition

$$\text{True } B \equiv B \qquad (4),$$

though not necessarily its implicational strengthening

$$\text{True } B \leftrightarrow B \qquad (5).$$

Now by paradox arguments an assertion C can be found such that $C \leftrightarrow \sim\text{True } C$, whence $C \,\&\, \sim C$. Thus truth is expressible at the cost, on the dialectical picture, of isolated contradictions. The dialectical picture allows, then, for both classically admissible content self-dependence which leads classically to undecidability and classically inadmissible content self-dependence which leads to inconsistency. And this is as it should be: both sorts of cases are out of the same box, and should be admitted, or rejected, together.

To adopt, however, a dialectical metalanguage as well as (or together with) a dialectical object language is to venture on to more slippery ground than will be attempted in the sections that follow. For given the vanishing theory of truth, as expressed by (5), and the law of non-contradiction, $\sim(A \,\&\, \sim A)$, both Aristotle's and Hegel's theses emerge. But these theses - respectively, "for no assertion A is both A and ~A true", and "for some assertion B, both B and ~B are true" - are usually regarded as totally antithetical. That they should both hold in a strong dialectical metalanguage is merely a reflection, by way of (5), of the object theses, $\sim(A \,\&\, \sim A)$ together with the thesis, for some B, $B \,\&\, \sim B$, of dialectical logic.

Ascent to a dialectical metalanguage emphasizes that there are two stages in a more complete switch from classical to dialectical positions, and that it can be important pedagogically (and in order to keep one's feet on the ground) to separate the stages. The first is the switch to a dialectical logic, where the semantical metalanguage is kept classical. In [8] and in most of this paper only this switch is attempted. The second, and eventually required, conversion is that of the semantical metalanguage to a dialectical one. The second stage is required for a full dialectical assessment of limitative theorems, and also, to take a simple example, for dialectical evaluation of the vanishing truth principle, given the inclusion in orthodox relevant logics of the law of non-contradiction $\sim(A \,\&\, \sim A)$. If (5) is incorporated then the two positions which stand in dialectical contrast at the first stage, the Aristotelian and the Hegelian positions, are synthesized. While the synthesis has its appeal, there are good reasons for avoiding it, for dropping (5) and making the truth connective do some proper work, not vanishing.

§6. Dialectical set theory. A dialectical set theory is one which accepts the paradoxes of set theory as part of the theory; it is a theory on which the underdeterminacy and overdeterminacy induced by the paradox-generating items of the set-theoretical paradoxes is simply admitted and the paradox arguments are taken as proofs; it is a theory according to which the Russell class, for example, the class

of all those classes which are not self-membered, both does and also does not belong to itself, and thus is perforce an inconsistent theory. But the hope (which can be vindicated under certain conditions) is that it is not a trivial theory on which not just the Russell paradox, but everything, holds.

A dialectical set theory, then, meets the paradoxes head-on, taking the paradox arguments as proofs and the contradictory conclusions as holding in the theory. That this approach has hardly ever been considered in modern discussions of the options open in the foundations of mathematics (or, if considered, quickly dismissed as crazy or, more restrainedly, as absurd or incoherent) is, once again, because it cannot be admitted classically.

To take the apparently radical - but in fact commonsense - position that some sets (or multiplicities, if you like) are inconsistent and just do involve contradictions, one need not, however, embrace contradictions as true. By suggesting that such a position would admit contradictions as true, Hilbert and Ackermann ([17], p.151) obliterate a most important distinction, namely that between theories that are simply considered and those that are taken as true. A theory can be investigated logically without being taken to be true; indeed it may be known to be false. For logical investigation of dialectical theory to get under way it is enough to have it conceded - what surely ought to be conceded - that there are non-trivial inconsistent theories, and that set theory may be one of these. But - according to the more orthodox option open here - since contradictions hold in the theory, the theory cannot be true, the world of set theory must be different from the actual world. In what follows this option, that unreconstructed set theory is a non-trivial inconsistent and false theory, will not be ruled out.

But there is a more exciting option, the thorough-going dialectical position, according to which unreconstituted set theory is an inconsistent but nonetheless true theory. Set theory as originally developed by Cantor is such a theory, and Cantor can be construed as taking it as such a theory, as I shall try to bring out. Subsequent set theories, starting with Russell and Zermelo, have all been attempts to consistencize in one way or another the inconsistent, but still thoroughly appealing, Cantorian theory, by dropping apparently true statements from the theory, e.g. one or other or both of the statements that the Russell class R belongs to itself and that R does not belong to itself.[7] The dialectical position is however that there is nothing wrong with the original unformalised Cantorian theory: disaster only occurred with the formalisation of theory which underpinned it with classical logic and thus trivialised it.

Cantor's set theory is not alone in being a promising candidate for an inconsistent but correct theory. As we have observed, if empirical facts are unfavourable so that the conditions for various semantical paradoxes are satisfied then ordinary discourse (considered

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[7] Meyer's way of consistencizing a theory (in [60]) through a converse Lindenbaum lemma illustrates a more general method of separating out a crucial consistent part of an inconsistent theory.

as a theory) is inconsistent (e.g. the statements of the policeman and the prisoner in one of Prior's family of paradoxes [9] really do generate contradiction). Once again, as with the logical paradoxes, there are many ways - only a few of them worked out in any detail at all - of rendering discourse consistent - each of them deleting or making inexpressible apparent truths.

The thoroughgoing dialectical position - unlike the more orthodox option (which is however hardly a comfortable position in the case of semantical paradoxes), has the shocking consequences of making the world (considered as everything that is the case) inconsistent and of making contradictions possible, since what is true is possible. As argued in [8], a metaphysical position that does these shocking things cannot be automatically ruled out. The consistency of the world, when one thinks about it, is not at all easy, and perhaps impossible, to establish in a non-question begging way. The objection, for example, that if this were so contradictions would be possible - but they are not, begs the question by conflating two senses of 'possible', namely entailing a contradiction, and being realisable, which a careful dialectician would distinguish. Good arguments in favour of the consistency assumption, as distinct from prejudice, are hard to come by.

Perhaps, but why choose a dialectical theory for the paradoxes? The reasons are as before. A first major reason for going dialectical is this:- Logical reasoning does not simply cut out when one lands in a paradoxical situation - as one easily may: just consider again some of the families of paradox Prior [9] discusses. Nor does catastrophic breakdown occur as the classical account presumes (so the classical view might well be called the catastrophe view of inconsistency and paradox). For example, the policeman-prisoner paradoxical situation could occur (with the policeman saying that everything the prisoner says is false and otherwise making only independently verifiable true statements while the prisoner simply says that whatever the policeman says is true), yet courtroom procedure would most likely go on much as before. And the logical paradoxes are far more remote. Certainly even if there were some breakdown in a real life paradoxical situation it would generally be isolated and insulated from most other matters. But, whatever the limited breakdown, if any, reasoning would not come to a sudden halt. On the contrary, it has to be applied to spell out the consequences drawn in the paradoxes from the often unsurprising (at least to the uninitiated) premisses.

It is not just that we want to apply logical principles in the region surrounding the paradoxes, the paradoxes themselves being out of logical bounds or taboo. It is that we want to apply logical principles to, and right inside, the paradoxes. Just as we want to be able to reason in and find our way about logically inconsistent situations, so likewise in the case of paradoxical situations. Even inside the paradoxes, in paradoxical situations, we can continue to reason. These situations have a logic - but it is severely non-classical - and they have a logic it would be desirable to determine logically. This cannot be done if paradoxical situations remain a prohibited area. Thus any proposed resolution of the paradoxes which aims to get them out of the way before logic can apply will be insufficiently general; a satisfactory resolution should enable the

study of paradoxicalness and the workings of the paradoxes by logical techniques. These points tell particularly against type-theoretic blockings of logical paradoxes and levels-of-language strictures.

These are not of course the only reasons for favouring a dialectical theory. The paradoxes would not be paradoxes were there not a convincing, appealing or even compelling case for the assumptions they involve. In the case of the logical paradoxes the arguments are not just those of an inconsistent logical theory; they derive from assumptions which are commonly thought to be true (in common parlance logical paradoxes are "true contradictions"). This is especially so in the case of the comprehension scheme for set characterisation, which appears to be the main source of a striking array of set-theoretical paradoxes. For a comprehension principle is built into the meaning of set.

Going dialectical is only a part of the paradoxical story. Another part consists in distinguishing the various everyday notions of set that have been conflated with set in the pure abstract sense (characterised by comprehension and extensionality); and yet another major part lies in providing the mechanisms of the paradoxes. Both the latter and the matter of determining the non-paradoxical sets can be done, I believe, through a theory of loops. In the case of paradoxical sets there is always some assertion where there is no end to looping. Both dialectical options indicated require supplementation by some account which marks out the non-paradoxical sets, but the matter is more critical with the weaker option which should be able to go on to say what is correct. In the stronger option this only amounts to working out what is non-paradoxically true, on the various consistent cut-downs—which may not be taken to be of such great or immediate moment.

Let us simplify matters, this time in a conventional manner, by restricting considerations to first-order set theories, that is to set theories formed by adding to quantificational logic only constant predicates concerned with sets themselves, e.g., ϵ, $=$, M (for 'is a set' or 'is a multiplicity'), E (for 'is an ensemble' or 'is a consistent multiplicity'). The theory will of course always include ϵ and be able to define $=$. A simple (first-order) dialectical set theory DST will accordingly comprise some non-classical quantificational logic with at least a conventional set of connectives $\{\&, \vee, \sim, \rightarrow\}$ and universal and particular quantifiers, U and P, with denumerable stocks of subject variables and perhaps constants, and with some predicate constants including ϵ. The formation rules of DST will be like those for other first-order set theories. The postulates of DST will be those of its non-classical quantificational logic together with some characteristic set-theoretic postulates constraining the predicate constants.

The critical issue - which reflects back on the choice of the quantificational logic - concerns the shape of the set theoretic axioms, particularly the versions of comprehension and extensionality adopted. Let us consider comprehension first. DST does not seek to avoid the paradoxes, so it does not need to write in the usual restrictions on comprehension designed to secure consistency. On the contrary, DST aims to incorporate the paradoxes and the reasoning involved in them. Indeed, as a further condition of adequacy, DST

has to be able to establish that the Russell class R both belongs to itself and not, i.e. $R \in R \ \& \sim(R \in R)$. The argument for this relies on an application of the comprehension axiom of the form

$$(x)(Pw)(x \in w \text{ iff } \sim(x \in x)),$$

a form ruled out on all going set theories except ideal or naive set theory. The immediately suggested idea, which has tempted many, is: if such forms are admitted why not the original comprehension axiom of ideal set theory itself? After all no one has been able to find anything very convincing wrong with the comprehension axiom, except things bound up with the paradoxes themselves. But DST is already committed to admitting the paradoxes, so why restrict comprehension at all?

But it is, of course, no convincing argument for the comprehension axiom that no one really has anything against it - except paradoxes - and that the naive form is still bought fairly generally in workaday mathematics and in everyday reasoning. The axiom may nonetheless have to be faulted because it trivialises, and even a dialectician is going to be forced to restrict comprehension if he makes the mistake of adopting an over-restrictive sentential logic. No, the appeals to ordinary unrestricted usage of comprehension in set formation are persuasive, because they reflect the fact that set formation is not limited by any restrictions at all. Every condition - whether intensional, paradoxical, or whatever - determines a set through a comprehension principle.

Accordingly the next adequacy requirement imposed is that DST admit an unrestricted comprehension axiom, namely

$$(x)(Pw)(x \in w \leftrightarrow A) \qquad \text{(GCA)}$$

where $\leftrightarrow$ is the iff connective of DSL (which is here taken as defined $B \leftrightarrow C =_{Df} (B \rightarrow C) \ \& \ (C \rightarrow B)$) and A is any well-formed expression of DST.

The comprehension axiom is a general one (whence the label GCA) because it does not impose the familiar restriction, that w should not be free in A, that even naive set theories often adopt. Removing this substitutional restriction opens the way for the formation of further inconsistent sets, e.g. most simply, a set Z with the property that

$x \in Z \leftrightarrow \sim x \in Z$, upon writing $\sim x \in Z$ for A in GCA.

Z, unlike R, is a completely bizarre set, everything belonging to it iff it does not, whereas very many sets do not belong to R without also belonging, namely all those that are straightforwardly non-self-membered such as the set of all integers, the set of all purple items, all concrete objects, etc. If the dialectician is going to tolerate some inconsistent sets isolated further inconsistent sets might as well be admitted as well, especially if there are reasons and advantages in doing so. In this case there are. A major uniformity in set determination can be achieved, through the derivation of such

postulates as the axiom of choice from GCA, as we shall see. It may be objected that with the GCA we lose the constructive character of set generation that the CA provides, with sets always eliminable through their characterising condition - whereas Z for example is deliberately characterised in terms of itself. But this constructive character of CA was always a myth, and the paradoxes show that eliminability fails (in cases where looping occurs). Constructive generation of sets from some initially given base leads to theories of constructive sets in the vicinity of (not necessarily coinciding with) Zermelo's set theory. It does not accord with the usual or Cantorean view of sets - as given. If sets are all out there, in Aussersein, as they seem to be, then any constructive aspect vanishes. A GCA provides a general, and legitimate, method of picking them out. Finally there is of course no reason why we should not, within the theory, distinguish sets by their method of generation just as we may distinguish consistent and inconsistent sets; thus, e.g. we might distinguish CA sets from those like choice sets that use GCA, we might distinguish constructive sets, and so on.

Some fairly severe constraints on the character of the non-classical logic of DST are already imposed by the conditions of adequacy adopted. For example, the requirement that the Russell paradox $R \in R \;\&\; \sim(R \in R)$ be derivable requires U-instantiation, and enough principles to guarantee the inference from $R \in R \leftrightarrow \sim(R \in R)$ to $R \in R \;\&\; \sim(R \in R)$, the set I shall adopt being the sufficiency conditions: Identity ($A \to A$), Simplification ($A \;\&\; B \to A$, $A \;\&\; B \to B$), Modus Ponens (A, $A \to B \twoheadrightarrow B$), Adjunction ($A$, $B \twoheadrightarrow A \;\&\; B$), Excluded Middle ($A \vee \sim A$) and $\vee$-Composition ($A \to C \;\&\; B \to C \to .\; A \vee B \to C$). This set is of course chosen with several ulterior motives, one of them being the retention of a single universal logic. Then the argument proceeds as follows:-

1.	$R \in R \to \sim(R \in R)$	Simplification
2.	$\sim(R \in R) \to R \in R$	
3.	$R \in R \to R \in R$	Identity
4.	$R \in R \vee \sim(R \in R) \to .\; R \in R$	by 2, 3, $\vee$-Composition, and rules
5.	$R \in R$	4, Excluded Middle
6.	$\sim(R \in R) \to \sim(R \in R)$	Identity
7.	$R \in R \vee \sim(R \in R) \to \sim(R \in R)$	, as for 4
8.	$\sim(R \in R)$	, as for 5
9.	$R \in R \;\&\; \sim(R \in R)$	, Adjunction.

The most controversial of these principles is very likely Excluded Middle: its adoption in DST is only the first of many features that serve to distinguish the dialectical approach from a many-valued approach to set theory. Adoption of Excluded Middle in this framework is tantamount to adoption of a rule form of Reduction, viz. $A \to \sim A \twoheadrightarrow \sim A$ and, given Contraposition, of Counterexample, viz. $A \to B \twoheadrightarrow \sim A \vee B$. The many-valued alternative, against which I shall argue, though accepting $R \in R \leftrightarrow \sim(R \in R)$ rejects $R \in R \;\&\; \sim(R \in R)$, the separation being effected by the abandonment of Reductio principles and Excluded Middle. But the question, as to whether R belongs to

itself or not, is (as argued elsewhere, e.g. SC, [20]) a perfectly significant one; and the trouble is not that the truth-value of $R \in R$ is indeterminate - the trouble is that it is overdetermined, so that both $R \in R$ and $\sim(R \in R)$ should, on quite compelling intuitive grounds, get assigned value true. Dialectically abandonment of Excluded Middle is the wrong option. Dualising the principles so far adopted takes us well on the way to the logical principles we shall eventually and tentatively adopt for DST.

Much more sentential logic is quickly ruled out than is easily ruled in. (This is of course the trade-off we depend upon - a strong non-trivial set theory by way of weak sentential principles which don't wipe out reasoning within and through the paradoxes.) Firstly, as we have observed all the paradoxes which spread contradictions about go, e.g. $A \rightarrow .\ \sim A \rightarrow B$, $\sim A \ \& \ A \rightarrow B$, and so on. They're no-gooders which we'd want to get rid of on other grounds anyway, though doing so already makes the theory severely non-classical. More disconcerting classically however is the elimination of the rule γ of material detachment. But it is bound to go. By the dual of Simplification, namely Addition, $\sim(R \in R) \vee Bad$; so, were γ to hold, Bad would ensue, for arbitrary Bad, i.e. γ would trivialise DST. (The dialectical case against γ is set out in DL: γ holds at best for consistent theories.)

The rejections are not disconcerting just for the classically-oriented. The philosopher who is beginning to be charmed by one of the "standard" systems of relevant logic, of ABE, will perhaps be alarmed, if not by the disappearance of Contraction $A \rightarrow (A \rightarrow B) \rightarrow .\ A \rightarrow B$, at least by the rejection of its mate Assertion, $A \ \& \ (A \rightarrow B) \rightarrow B$. Assertion would trivialise DST because it leads to a rule form of Contraction, and this rule form enables a proof of triviality through the Curry paradox argument,[8] as follows:-

$(Pw)(x)(x \in w \leftrightarrow .\ x \in x \rightarrow Bad)$ by CA.

Let us call such a w, C; the quantificational principles of DST should be chosen to permit this procedure. Then

$(x)(x \in C \leftrightarrow .\ x \in x \rightarrow Bad)$; so

$C \in C \leftrightarrow .\ C \in C \rightarrow Bad$, by Instantiation.

Thus $C \in C \rightarrow .\ C \in C \rightarrow Bad$ and

$C \in C \rightarrow Bad \rightarrow .\ C \in C$, upon simplifying.

By Rule Contraction, $C \in C \rightarrow Bad$, whence $C \in C$, so Bad ensues. Rule Contraction is obtained in this way:

$A \rightarrow (A \rightarrow B) \dashv\vdash A \ \& \ A \rightarrow .\ A \ \& \ (A \rightarrow B)$, by the dual of $\vee$-Composition,

$\dashv\vdash A \rightarrow B$, using Tautology $(A \rightarrow .\ A \ \& \ A)$,

Rule Syllogism ($A \rightarrow B$, $B \rightarrow C \dashv\vdash A \rightarrow C$), and Assertion. Rule Syllogism (already adopted earlier) we'd be extremely reluctant to give up (for reasons set out in RLR), Tautology we're committed to by the dual of

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[8] This trivialisation was first observed by Meyer.

∨-Composition and Identity, so Assertion has to give. But there are in fact appealing reasons drawn from investigations of semantical paradoxes, for thinking that Assertion - which is quite different from the rule of Modus Ponens - is a pretty dubious customer once paradoxical situations are admitted. This is best brought out semantically: Assertion would exclude situations, of the type which occur with semantical paradoxes, where $A \to B$ and A both hold but B fails to hold, that is, where an implication which holds is also counterexampled.

The rejection of Contraction principles eliminates not only standard relevant logics but all logics based on positive logic - in particular, Jaśkowski's system and da Costa's C_n systems ($1 \leqslant n \leqslant w$), all of which were specifically fashioned to study, in one way or another, systemic inconsistency (see, e.g., [35] and [36]).

The remaining considerations determining choice of a relevant working logic to underpin dialectical set-theory have been semantical ones, with two exceptions. In particular, the Affixing rules adopted have been chosen with a view to ensuring simple semantical modellings with a 3-place relation R defined on worlds (or situations); for without their adoption a more complex relation on worlds and sets of worlds would be required. And simplicity of modelling is important in attempting to establish results concerning DST, such as absolute consistency. But nothing much of importance hangs on the Affixing rules, and were they to become essentially involved in the derivation of underivable results they could be abandoned without excessive hardship. Likewise in the case of the remaining quantificational schemes, just enough to retain a simple rigid semantics and theses required for a completeness argument has been included. The exceptions lie in the strengthenings to theorem form of the rules of Syllogism and Contraposition. Both schemes are correct and belong to the basic system DL (of [8]). Conjunctive Syllogism, $(A \to B) \,\&\, (B \to C) \to .\ A \to C$, does however introduce a modelling condition it would be preferable to do without; but its use is required in the expected development of DST.

The resulting quantification logic DKQ, with ∨ and P defined operators, is as follows:-

$A \to A$	$A \,\&\, B \to A$
$A \,\&\, B \to B$	$(A \to B) \,\&\, (A \to C) \to .\ A \to B \,\&\, C$
$A \,\&\, (B \vee C) \to .\ (A \,\&\, B) \vee C$	$(A \to B) \,\&\, (B \to C) \to .\ A \to C$
$\sim\sim A \to A$	$A \to \sim B \to .\ B \to \sim A$
$A \vee \sim A$	
$(x)A \to A(t/x)$	$(x)(A \to B) \to .\ A \to (x)B$, with x not free in A.
$(x)(A \vee B) \to .\ A \vee (x)B$, with x not free in A.	
$A,\ A \to B \to\!\!\!\circ\ B$	$A,\ B \to\!\!\!\circ\ A \,\&\, B$
$A \to B,\ C \to D \to\!\!\!\circ\ B \to C \to .\ A \to D$	$A \to\!\!\!\circ\ (x)A$

The sentential logic DK of DKQ differs from system DL only in weakening Counterexample, $A \,\&\, B \to \sim(A \to \sim B)$, to rule form, that is, in effect, to $A \vee \sim A$.

The first degree structure of DKQ, that is to say of the system of wff which contain no nested occurrences of the implication $\rightarrow$, is exactly that of such better known relevant systems as EQ and RQ. The comparative loss of strength of DKQ is where it ought not to matter so much for formalising intuitive arguments, in iterated implicational principles. Note too that all but the principles for implication of DKQ are essentially classical: unlike most proposals for antinomic logics in the literature, negation satisfies all forms of double negation, contraposition, and non-contradiction.

The applied system DST results by adding to DKQ the general comprehension scheme GCA and an axiom of extensionality. Evidently for but a little care in the choice of sentential axioms a great saving in set theoretical superstructure has been achieved.

DKQ is not a finitely-valued logic: there are no finite characteristic matrices for the sentential part of DKQ (of which the quantificational part is a conservative extension). This important fact separates the dialectical approach from many-valued approaches to the set-theoretical paradoxes, but it is not the critical separation point. The critical separation point which equally separates DST from infinite-valued approaches, e.g. those based on Łukasiewicz infinite-valued logic L_∞, is the treatment of over-determined cases. The usual non-standard logic approach has been to treat over-determined cases like under-determined cases, the truth-teller platitude like the Liar (or false-teller) paradox, the anti-Russell class $\breve{R}$ (of all those classes which are self-membered) like the Russell class, and so on. This is wrong. There is a symmetry between over-determined, or inconsistent, and under-determined, or incomplete, cases but they do not reduce to one another. (This is also a reason why significance-style solutions are mistaken, though not the main reason.)

A theory like DST which mirrors the actual situation will, then, be both inconsistent and incomplete. It will not assign a truth-value to $\breve{R} \in \breve{R}$ for example, though it could of course easily be extended to do so. That is not to say that $\breve{R} \in \breve{R}$ does not have a truth-value - by bivalence in the actual world, T, $\breve{R} \in \breve{R}$ is either true or false - but nothing in the theory, like nothing in ordinary intuitive considerations, determines which, so it seems.

§7. The problem of extensionality and of relevant identity. To determine DST it remains to specify the form of the extensionality axiom for sets. The form of this axiom raises an awkward problem not just for DST but for intensional set theory generally. The problem can usefully be broken down into two subproblems.

The first problem is how to formulate the presumably analytic claim

A. Sets with the same elements are identical. This claim is the standard way of taking up the thesis that sets are determined by their elements (which is what supposedly distinguishes sets from properties). Claim A can be transformed, without too much dispute, to the determinable form

A1. If $(x)(x \in w \text{ iff } x \in v)$ then w is identical with v, w=v for short. The question is how to fill out the determinables 'iff' 'if ..., then' and '=' with appropriate determinates (on this method of resolving disagreements into choice of determinates for agreed determinables, see SC, chapter 4). Some resolutions of determinables are obviously unsatisfactory in an intensional framework, e.g. the classical formulation

$$(x)(x \in w \equiv x \in v) \supset .\ w=v;$$

for though w and v may coincide everywhere in the actual situation, they may well differ in situations beyond the actual; so their element coincidence would at best guarantee an extensional (contingent) identity, and certainly not an identity of the sort required for intersubstitutivity of the kind "extensionality" axioms demand. This trouble can be avoided by strengthening the material equivalence to a coentailment, which ensures coincidence through all worlds considered. Within a relevant dialectical framework, where material detachment is not available, the most promising resulting forms are

A2. $(x)(x \in w \leftrightarrow x \in v) \rightarrow .\ w=v$, and

A3. $(x)(x \in w \leftrightarrow x \in v) \twoheadrightarrow .\ w=v$.

The way the identity determinable, =, is to be resolved takes us to the main problem, the characterisation of identity relevantly - and also intensionally - and the question

B. Within what frames are identicals interchangeable?

The answer conventionally assumed in set theory is: within all frames. Thus results the determinable form:

B1. if w=v then A(w) iff A(v).

If, however, the determinables are resolved in the expected way so that B1 becomes

B2. $w=v \rightarrow .\ A(w) \leftrightarrow A(v)$

then serious problems ensue: Firstly, irrelevance. This results immediately from B2 by vacuous substitution. Thus $w=v \rightarrow .\ p \rightarrow p$ for arbitrary wff p, e.g. $2 + 2 = 4$ entails that GCA implies itself. Not only is this irrelevant in an obvious sense, that antecedent and consequent have little or nothing to do with one another, but further the antecedent is insufficient for the consequent; and evidently an arbitrary arithmetic identity is not sufficient on its own for all instances of the law of identity. To say that it is is nothing but a restricted paradox of implication. The problem cannot be escaped simply by prohibiting vacuous substitutions; for, as both Bacon and Dunn have observed, since $B \leftrightarrow .\ B \,\&\, (B \vee A(w))$, cases of vacuous substitutions can be quickly derived from cases of non-vacuous substitution. B2 will have to be more drastically modified to avoid irrelevance. (The situation with regard to orthodox entailmentally strengthened identity theory in the framework of stronger relevant logics is even worse. For there it can be shown that any identity entails any logical truth.) Secondly, there are familiar problems of opacity

where highly intensional frames are admitted into the language (see, e.g., Quine [10] for the problems, and for the treatment of the problems SC, chapter 7). This latter problem is not avoided, but only shifted, by different resolutions of the determinables of B1. In the absence of comprehension principles both problems are however avoided (as Urquhart first saw) by replacing B1 and B2 by the initial forms from which inductive arguments for them begin, namely:

B3. $w=v$ only if $w \in z$ iff $v \in z$ and

B4. $w=v \rightarrow .\ w \in z \leftrightarrow v \in z$.

Within a weak relevance framework there are, in fact, two resolutions of B2 which are promising,

B5. $w=v \rightarrow .\ w \in z \rightarrow v \in z$ or, what is deductively equivalent,
$w=v \rightarrow .\ (z)(w \in z \rightarrow v \in z)$, and the corresponding rule form

B6. $w=v \twoheadrightarrow (z)(w \in z \rightarrow v \in z)$.

And B2 is not recoverable from B5 by inductive argument since the relevant logics being considered all lack the principle of Factor (e.g. $A \rightarrow B \rightarrow .\ C \,\&\, A \rightarrow C \,\&\, B$).

It would be rash to pretend however that this avoidance of the problems by moving back to primitive or atomic forms is philosophically satisfactory: since from a natural language point of view any predicate, including highly opaque predicates, can be taken as primitive, the move involves an unfortunate logical atomism. And when general comprehension principles are added the unsatisfactoriness of this atomic fix becomes formally transparent. For consider an arbitrary wff A(w) containing w free. By the comprehension principles for some set z, say z_1, $w \in z_1 \rightarrow A(w)$, for every w, including v. So B3 yields B2 by quantificational principles. The upshot is: general comprehension and extensionality principles cannot be combined without sacrificing relevance.

The conflict between comprehension and extensionality is part of a general quandary for relevance logics that is quite independent of logical paradoxes, namely that substitution or, what are closely connected, comprehension principles cannot be relevantly coupled with Leibnitz identity, or extensionality principles. The problem occurs even for second order quantified relevant logics where logical paradoxes are of course no threat.

Of the three possible ways out of this quandary, qualify Leibnitz identity, qualify comprehension (or substitution), and qualify both, the second is ruled out given quite weak relevant quantificational principles, as the earlier argument to identity irrelevance reveals. The damaging effect of extensionality principles is readily shown, e.g. by the following argument:

Since $p \leftrightarrow q \rightarrow .\ f(p) \rightarrow f(q)$, by extensionality,
$p \leftrightarrow q \rightarrow .\ r \rightarrow r$, taking f(p) as constant r.

So, as $p \to p \to$, $p \leftrightarrow p$, $p \to p \to$, $r \to r$. That is, total irrelevance follows from extensionality. Nor is the matter rectified by the simple expedient of excluding vacuous substitution. For vacuous substitution can be reintroduced by the back door given the mere exclusion of vacuous substitution, as the earlier argument for identity irrelevance revealed. Just how damaging extensionality principles can be is brought out, unintentionally, by Church in [11], where classical sentential logic is derived, from otherwise largely innocuous postulates by way of extensionality.

The Belnap-Meyer strategy has been to try to qualify both extensionality and comprehension, the main proposal so far being that these principles operate only where (sentential) functions depend relevantly on their arguments. Thus in both $w=u \to A(w) \to A(u)$ and $(Pz)(w)(w \in z \leftrightarrow A(w))$ it is required that $A(w)$ depends relevantly on (or really is a function of) w. For example, constant functions (such as t in the sentential case) do not so depend on their arguments. But the notion of relevant dependence so far lacks a satisfactory explication; all that has been offered in the case of identity is the unsatisfactory atomic scheme B5 already considered (starting from which dependence is defined inductively in the obvious way which corresponds to inductive recovery of a qualified B2).

If, however, extensionality has to be qualified anyway - and there is no independent case for qualifying comprehension - a better course (also recommended, for what it is worth, by the minimum mutilation principle) is to qualify extensionality alone, and thus to reject the Leibnitzian account of identity. As to how this is to be done the form of the content identity principle in the sentential case gives a good clue. This coimplication substitutivity principle takes the rule form:

$$A \leftrightarrow B \multimap D(A) \to D(B) :$$

strengthening it to entailment strength (with an arrow replacing the rule connection) generates irrelevance.[9] Proceeding analogously in the set theoretical case leads to B6,

$$w=v \multimap w \in z \to v \in z \qquad \text{(Ext. R)}.$$

In the case of set theory the problem as to how to guarantee the appropriate properties of intensional identity is easily solved: simply define (as in standard set theories)

$$(D=)\ w=v =_{Df} (x)(x \in w \leftrightarrow x \in v) ,$$

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[9] There are other options to consider, the most important being the form: $A \leftrightarrow B \ \& \ D(A) \to D(B)$. There are reasons for having serious reservations about this principle - reasons based on a consideration of its inductive proof. Firstly, the inductive base $A \ \& \ (A \to B) \ \& \ (B \to A) \to B$ seems to depend for its proof and truth on the rejected Assertion principle. Secondly, the inductive step for negation would require the use of Antilogism: $A \ \& \ B \to C \to$, $A \ \& \ \sim C \to \sim B$, which is an immediate source of paradox and of Disjunctive Syllogism.

i.e. use A2 and its converse to characterise set identity.[10] It is immediate from the definition, using quantificational principles, that identity, =, is reflexive, symmetric and transitive,[11] These properties are used in turn in establishing, by induction, the general replacement rule:

$$w=v \multimap A(w) \rightarrow A(v) \ ,$$

where A(w) is any wff and A(v) results from A(w) by replacing v by w at one or more places, provided no occurrence of w is in the scope of quantifiers binding w or v.

(Ext. R), together with (D=), completes the weak axiomatisation of DST. The weak axiomatisation appears, however, to be too weak for some relevantly acceptable set-theoretic arguments, e.g. for some of those used below in arguing for the axiom of choice. The problem is that the rule form of extensionality does not capture all that was correct in the traditional formulation. A better approximation to what was right classically can be obtained (as so often) enthymematically using constant t. Then the extensionality principle becomes:

$$u=w \ \& \ t \rightarrow . \ u \in z \rightarrow w \in z. \qquad \text{(t-ed Ext)}$$

It follows by induction (since a t-form of factor holds, namely $A \rightarrow B \ \& \ t \rightarrow . \ A \ \& \ C \rightarrow B \ \& \ C$):

$u=w \ \& \ t \rightarrow . \ A(u) \rightarrow A(w)$, with appropriate quantificational qualifications. Since t is a theorem (Ext. R) follows at once; but irrelevance is seemingly avoided, since the extra t occurring in premisses is uneliminable.

To adopt this enthymematic approach to extensionality requires the addition of constant t to the underlying logic DLQ. But t - which is interpreted as the conjunction of all theorems, and semantically marks out the class of logically regular worlds - can be added conservatively to DLQ through the two-way rule: $A \multimap t \rightarrow A$. Constant t is worth getting to know. It has already played an important role in the algebraic study of relevant logic, and it may have a significant part in the properly relevant formulations of arithmetic and analysis.

But t-ed extensionality can in turn be improved upon. A superior form of extensionality - which can be gleaned from the corrected form of Factor, namely $A \leftrightarrow B \ \& \ C \leftrightarrow C \rightarrow . \ A \ \& \ C \leftrightarrow B \ \& \ C$ - is

$$u = w \ \& \ z = z \rightarrow . \ u \in z \rightarrow w \in z \qquad \text{(Ext)}.$$

This form, suggested by Brady, is stronger than t-ed Ext.

System DST proper has Ext as its final postulate.

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[10] The question arises as to how to interpret (D=) where w and v are not sets, but are, e.g., individuals. One possibility (Quine's idea for his system ML) is to interpret 'ϵ' as identity in such cases so that (D=) reduces to a form of $w=v \leftrightarrow . \ (x)(x=w \leftrightarrow x=v)$. A more interesting possibility is to interpret 'ϵ' as a part relation - so that w=v iff all parts of w and v everywhere coincide - since this opens the way for a neat amalgamation of set theory and mereology (a suggestion developed in Brady [12]).

[11] Proof of transitivity uses Conjunctive Syllogism.

§8. The development of dialectical set theory; reconstructing Cantor's theory of sets. Given comprehension and extensionality schemes it is an easy matter to develop most of the main features of classical set theory and extant descriptive set theory. The central postulates of Zermelo-Fraenkel (ZF) and von Neumann-Bernays-Gödel (NBG) set theories are readily derived - in intensional form - as a few illustrations will emphasize. Consider first a version of the familiar pairing axiom

(P) $(Pz)(u)(u \in z \leftrightarrow u=x \vee u=y)$.

This is an instance of GCA, and appropriate uniqueness of the pair set z so guaranteed follows from extensionality. For if $(u)(u \in z_1 \leftrightarrow u=x \vee u=y)$ and $(u)(u \in z_2 \leftrightarrow u=x \vee u=y)$ then $(u)(u \in z_1 \leftrightarrow u \in z_2)$, so by (Ext. R), $z_1=z_2$. Consider next the null set axiom:

(N) $(Pz)(u)(u \in z \leftrightarrow \sim(u=u))$,

again an instance of CA. Rule extensionality ensures that the set provided is appropriately unique. Call it, as usual, Λ. It follows that $(u) \sim u \in \Lambda$, as required. For its complement, V, defined as $\bar{\Lambda}$, $(u)(u \in V)$. CA of course guarantees an unrestricted complement for each set x, through

(C) $(Px)(u)(u \in z \leftrightarrow \sim(u \in x))$.

As a final example consider the power set axiom

(W) $(Pz)(u)(u \in z \leftrightarrow u \subseteq x)$,

where $u \subseteq x =_{Df} (y)(y \in u \leftrightarrow y \in x)$. Note that $\subseteq$ has the requisite partial order properties and is properly related to $=$. However $\subseteq$ differs significantly from the usual inclusion notion in that it does not follow that $\Lambda \subseteq x$: the usual inclusion of the null set in every set depends on a paradox of formal implication. But a different definition of Λ will yield $\Lambda \subseteq x$.

What is perhaps not quite so obvious is that DST furnishes important but more controversial axioms of set theory and settles major questions standard theories leave open. In particular, it delivers not only the axiom of infinity, but, more originally, the axiom of choice. Furthermore it puts us on the way to regaining the proof Cantor thought he had of the continuum hypothesis. Let us consider these issues in turn.

(1) The axiom of choice. The standard relations between versions of the axiom of choice and its conventional equivalents, such as well-ordering principles, maximal principles, etc.,[12] become problematic in an intensional framework. However several central, but perhaps non-equivalent forms follow in DST; just one example will be given. The derivation of all the forms depends crucially on GCA and the fact that the set variable, for the introduced set, may appear free in the set characterising formula.

AC1. There is a function f such that for every non-null set y, $f'y$ belongs to y; in precisifying symbols:

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[12] The extraordinary variety of extensionally equivalent forms are set out in Rubin and Rubin [52].

$(\hat{P}f)$ $(\mathrm{fnc}(f)\ \&\ (y)((Pz)(z \in y)\ \&\ t \to f`y \in y))$. Define a function as a univocal relation or a null one.

Consider the following thesis supplied by GCA:

(i) $(Pf)(x)(x \in f \leftrightarrow .\ (Pu', v')(x=\langle u', v'\rangle\ \&\ v' \in u')\ \&\ \mathrm{fnc}(f))$.

For such an f, $(Px)(x \in f) \to \mathrm{fnc}(f)$. But also, $\sim(Px)(x \in f) \to \mathrm{fnc}(f)$, since, by definition, if f is null it is a function. (The usual definition includes this case automatically, again by virtue of paradox.) Hence fnc(f), and also $t \to \mathrm{fnc}(f)$. Now

(ii) $\mathrm{fnc}(f)\ \&\ (Pz)(z \in y) \to yf(f`y)$;

this can be made a matter of characterisation of $f`y$. Hence

(iii) $(Pz)(z \in y)\ \&\ t \to (\langle y, f`y \rangle \in f)\ \&\ t$.

To simplify (i) appropriately one further background detail is needed, namely

$\langle u, v\rangle = \langle u', v'\rangle \leftrightarrow .\ u=u'\ \&\ v=v'$.

Then, it follows from (i)

$(u, v)\ .\ \langle u, v\rangle \in f\ \&\ t \to .\ v \in u$;

so

(iv) $\langle y, f`y\rangle \in f\ \&\ t \to f`y \in y$.

Thus by (iii) and (iv) $(y)((Px)(z \in y)\ \&\ t \to f`y \in y)$, as required.

(2) Inconsistent and consistent sets. Just as there is nothing new about the idea of a dialectical set theory on its own, so there is nothing new about the inconsistent sets that such a theory will generate. The distinction between inconsistent and consistent multiplicities, aggregates or classes, dates back at least to Schröder ([54], p. 213) who, before the dark era of the logical paradoxes, marked out a set as consistent when its elements are compatible, and as inconsistent otherwise. Cantor's important distinction is more complex. On his initial account in his letters to Dedekind ([24], p. 114) an inconsistent multiplicity is one

> such that the assumption that all its elements "are together" leads to a contradiction, so that it is impossible to conceive of the multiplicity as a unity, as "one finished thing". ... If on the other hand the totality of the elements of a multiplicity can be thought of without contradiction as "being together", so that they can be gathered together into "one thing", I call it a consistent multiplicity or a "set".

Cantor's characterisation is hard to come to grips with formally,[13] since many of the notions used do not figure in elementary class theory, and since it is unclear from the larger context whether the contradictions of inconsistent multiplicities are merely hypothetical ones or are genuine, that is to say whether inconsistent multiplicities really are inconsistent in some respect. The classical interpretation, for which there is some textual support, is the hypothetical one: inconsistent multiplicities, though definite, are not really inconsistent. This leaves much to be explained that Cantor never explains, in particular as to how the comprehension axiom is to be restricted so that inconsistent multiplicities remain at most hypothetically inconsistent, and which multiplicities are consistent. It is worth exploring an alternative dialectical interpretation where none of this explaining is needed.

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[13] There are, however, several valuable leads in the literature. Firstly, Cantor's inconsistent multiplicities are strikingly similar to Russell's self-reproductive classes, classes which once formed generate further elements beyond, but not beyond, the class:

> ... there are some properties such that, given any class of terms all having such a property, we can always define a new term also having the property in question. Hence we can never collect <u>all</u> the terms having the said property into a whole; because whenever we hope to have them all, the collection which we have immediately proceeds to generate a new term also having the said property ([55], p. 36).

Russell goes on to claim that this provides the general form of all known class paradoxes, and that these paradoxes all belong to logic and are (only) to be solved by some change in current logical assumptions. He then considers three alternative changes in the comprehension principle designed to <u>exclude</u> self-reproductive classes, neglecting entirely the dialectical option of modifying other going logical assumptions. All Cantor's examples of inconsistent multiplicities are, or are intended to be, of the self-reproductive variety, but he does not aim to exclude them, and the most Russell has shown is that we cannot collect the elements of such classes into <u>consistent</u> wholes. Consider Cantor's first example, the "totality of everything thinkable". When this class is formed, or comprehended, it provides a further thinkable, which would not (Cantor must have supposed) belong to the initially formed class of thinkables and yet, with the comprehension of the class does belong to itself. Sethood, as ensured by the comprehension axiom, leads through self-reproduction to inconsistent properties.

Another important lead is provided by Ackermann's attempt [57] to decode Cantor's definition of 'set' and to build a set theory from the result. Ackermann's set theory can be seen as one way of determining precisely conditions on consistent sets, the predicate 'M', that is a distinctive feature of his system, reading: is a consistent multiplicity.

The central idea (see footnote 13) is that some sets are inconsistent because, when comprehension is applied, they turn out to have inconsistent properties. Thus, where x is a multiplicity, x is directly inconsistent iff $(Px)(x \in z \ \& \ x \notin z)$. But this definition does not, on its own, ensure that x is a multiplicity. By making use of extensionality, a superior definition, which implies multiplicity in the sense of having members, results:

$$x \text{ is d-inconsistent} =_{Df} (Pz)(z \in x \ \& \ z \notin x).$$

Then the Russell set, R, is d-inconsistent, since $R \in R$ and $R \notin R$. Similarly presumably, by paradox arguments, the sets consisting of all cardinals and of all ordinals are both d-inconsistent.

But direct inconsistency is insufficiently comprehensive. For if a set has an inconsistent part or an inconsistent element then presumably it is inconsistent in virtue of that. Cantor wants, in fact, to go further and to count a set as inconsistent if it can even be put into 1-1 correspondence with an inconsistent set. What needs to be guaranteed by appropriate definitions, then, are the postulates:

$$x \text{ is d-inconsistent} \rightarrow I(x), \text{ i.e. } x \text{ is inconsistent;}$$

$$I(x) \ \& \ (x \in y \vee x \subset y) \rightarrow I(y) .$$

(only the membership case causes a definitional problem, requiring introduction of the $\in$-ancestral.) It follows that the set U of all sets is inconsistent, since it has R as a part, and, what is more disputable, that the sets $\{R\}$, $\{\{R\}\}$, ..., are all inconsistent.

It is evident however that the important but laborious task of seeing what can, and, equally important, what cannot, be accomplished in DST lies, for the most part, ahead. What has been presented is but a small beginning in that enterprise.

§9. Ultramodal mathematics: arithmetic. In order to sustain the ultramodal challenge to classical logic it will have to be shown that even though leading features of classical logic and theories have been rejected, one can still get by. In particular, it will have to be shown that by going ultramodal one does not lose great chunks of the modern mathematical megalopolis. The strong ultramodal claim - not so far vindicated - is the expectedly brash one: we can do everything you can do, only better, and we can do more. More, because there are whole mathematical cities that have been closed off and partially abandoned because of the outbreak of isolated contradictions, notably theories of the very small, infinitesimals, and theories of the very large, Cantor's set theory. Admittedly there have been modern restorations of apparently consistent suburbs of these theories, but the life of these cities has vanished and they have become like modern restorations of ancient cities, like modern Balbus or Leptis Magna, mostly just patched-up ruins visited by tourists.

The question of the recovery of modern mathematics, or for that matter of classically-derivable mathematics, is not so far a very precise one, because there are different notions of recovery. In one

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ADDRESSES OF CONTRIBUTORS

Nuel D. Belnap, Jr.
Department of Philosophy
University of Pittsburgh
Pittsburgh, Pennsylvania 15260
U. S. A.

B. J. Copeland
84, James Street
Oxford, England

Harry Deutsch
1152½ North Fuller Avenue
Los Angeles, California 90046
U. S. A.

Michael A. McRobbie
Department of Philosophy
Institute of Advanced Studies
The Australian National University
Box 4, P. O.
Canberra, A. C. T. 2600
Australia

Richard Routley
Department of Philosophy
The Research School of Social Sciences
The Australian National University
Box 4, P. O.
Canberra, A. C. T. 2600
Australia

Robert G. Wolf
Department of Philosophical Studies
Southern Illinois University
Edwardsville, Illinois 62026
U. S. A.

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